Asymptotic Normality of Linear CDMA Multiuser Detection Outputs[1]

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Abstract — **This paper studies the asymptotic distribution of the output of a family of linear multiuser receivers. Every receiver in the family is a function of the channel correlation matrix of a simple form. It is proved that conditioned on almost all spreading codes, the output decision statistic of such a receiver converges to a Gaussian law in distribution as the system size goes to infinity. This set of receivers encompasses the matched filter, the MMSE receiver, the parallel interference cancelers, and many other linear receivers of interest. The normality principle solidifies the importance of the signal-to-interference ratio in the many-user analysis of linear multiuser detectors.**

The interest in the output distribution of multiuser receivers is deeply rooted in the analysis of the capabilities of coded and uncoded systems. For all but the decorrelator, the exact BER expression is a sum of an exponential number of Q-functions, the evaluation of which is infeasible for even moderately sized systems. To circumvent this difficulty, a Gaussian approximation of the multiple access interference (MAI) is often used, being rigorously justified in only very few cases, e.g., the MAI distribution of a matched filter converges to a Gaussian law in the sense of divergence [1], and the output MAI of the MMSE receiver is asymptotically Gaussian [2]. In this work, we prove asymptotic normality for a wide class of receivers. Our results encompass (in the equal-power case) the normality of the MMSE receiver shown in [2].

We assume an equal power, symbol-synchronous CDMA system where each user's spreading code is an independently and randomly chosen sequence of $+1$ and -1's. Consider a K-user system with a processing gain of N. The correlation matrix $\mathbf{R}$ is a $K\times K$ random matrix with R_{kj} as the crosscorrelation of user k and user j's spreading codes. Let the eigenvalue decomposition of $\mathbf{R}$ be $\mathbf{R}=\mathbf{U}\cdot\mathrm{diag}(\lambda_1,\cdots,\lambda_K)\cdot\mathbf{U}^{\mathrm{H}}$.

A linear receiver $\mathbf{G}$ is defined as a $K\times K$ matrix filter dependent on $\mathbf{R}$ that is applied to the matched-filter output. It outputs a vector of decision statistics expressed as

$$\mathbf{y}=\mathbf{G}\cdot(\mathbf{R}\mathbf{d}+\mathbf{n})=\mathbf{H}\cdot\mathbf{d}+\mathbf{z} \tag{1}$$

where $\mathbf{d}$ is a vector of symbols from the K users, $\mathbf{z}$ is a zero-mean Gaussian random vector with covariance matrix $\sigma^2\mathbf{G}\mathbf{R}\mathbf{G}^{\mathrm{H}}$, and $\mathbf{H}=\mathbf{G}\mathbf{R}$. We study a class of receivers taking the form of

$$\mathbf{G}=\mathbf{U}\cdot\mathrm{diag}(g(\lambda_1),\cdots,g(\lambda_K))\cdot\mathbf{U}^{\mathrm{H}} \tag{2}$$

for some continuous function g. If g degenerates to a constant, then $\mathbf{G}$ is reduced to the conventional receiver. If we let $g(\lambda)=(\lambda+\sigma^2)^{-1}$, $\mathbf{G}$ becomes the well-known MMSE receiver.

The channel input-output characteristic is given by the distribution of the decision statistic conditioned on the transmitted symbol. Without loss of generality, we focus on user 1's statistic (assuming $d_1=1$),

$$y_1=H_{11}+\sum_{k=2}^{K}H_{1k}d_k+z_1. \tag{3}$$

We study the limiting distribution of y_1 as the system size increases without bound, namely, K and N both tend to infinity with $K/N\to\beta\geq 0$. Note that every H_{1k} is dependent on K and N. We have the following result.

Theorem 1 *Conditioned on almost all spreading codes, the output statistic of a linear receiver* $\mathbf{G}$ *determined by continuous* g*, given by Eqn. (3), is asymptotically Gaussian in distribution . Its mean and variance are*

$$\mu_g = \int g(\lambda)\cdot\lambda\, dF_\Lambda(\lambda) \tag{4}$$

$$\sigma_g^2 = \int g^2(\lambda)\cdot\lambda\cdot(\lambda+\sigma^2)\, dF_\Lambda(\lambda)-\mu_g^2 \tag{5}$$

respectively. Here $F_\Lambda(\lambda)$ *is a probability distribution with a density function* $\sqrt{(\lambda-\lambda_{\min})(\lambda_{\max}-\lambda)}/(2\pi\beta\lambda)$ *on* $[\lambda_{\min},\lambda_{\max}]$ *and all its remaining mass at* 0*, where* $\lambda_{\min}=(1-\sqrt{\beta})^2$ *and* $\lambda_{\max}=(1+\sqrt{\beta})^2$.

The theorem is proved by first showing that the output statistic of every polynomial receiver defined as $\mathbf{G}=\sum_{i=1}^{m}x_i\mathbf{R}^{i-1}$ is asymptotically Gaussian in distribution by a central limit theorem. The following proposition, which has a non-trivial combinatorial proof, plays an important role.

Proposition 1 *For every positive integer* p*, and every user index* k*, the* p^{th} *order central moment of* $\sqrt{K}\left[\mathbf{R}^i\right]_{1k}$ *converges to a deterministic constant as* $K\to\infty$.

Asymptotic normality extends to any linear transformation $\mathbf{G}$ determined by a continuous function g, since it can be arbitrarily approximated by a sequence of polynomial receivers, whose outputs are all asymptotically Gaussian.

References

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[2] J. Zhang, E. K. P. Chong, and D. N. C. Tse, "Output MAI distribution of linear MMSE multiuser receivers in DS-CDMA systems," *IEEE Trans. Inform. Theory*, vol. 47, March 2001.

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