Minimum Probability of Error of Many-user CDMA Without Power Control[1]

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Abstract — **The minimum-probability-of-error (optimal) detector of multiuser CDMA is shown to be equivalent to a conditional mean estimator which finds the mean value of the stochastic output of a so-called Bayes retrochannel. The equivalence of the Bayes retrochannel to a spin glass allows the optimal bit-error-rate to be determined exactly in the large-system limit by studying macroscopic properties of the spin glass using tools developed in statistical mechanics. It is found that all users, regardless of their individual received energies, enjoy the same multiuser efficiency, which is directly related to the mean-square error of the conditional mean estimator.**

I. Introduction

This paper gives a summary of the results on uncoded error probability of multiuser detection for randomly spread CDMA in the many-user limit obtained in [1] applying concepts of statistical physics following the pioneering work of [2].

Consider a K-user Gaussian CDMA channel with a spreading factor of N described by

$$p(\mathbf{r}|\mathbf{d}_0,\mathbf{S};\sigma_0) = (2\pi\sigma_0^2)^{-N/2}\exp\left[-\|\mathbf{r}-\mathbf{S}\mathbf{A}\mathbf{d}_0\|^2/(2\sigma_0^2)\right] \quad (1)$$

where $\mathbf{d}_0$ is a vector of the K antipodal modulated symbols, $\mathbf{S}$ an $N\times K$ spreading matrix consisting random chips of ±1's, $\mathbf{A} = \mathrm{diag}(\sqrt{P_1},\dots,\sqrt{P_K})$ consisting of the received amplitudes, $\mathbf{r}_{N\times 1}$ the received signal vector, and σ_0 the noise variance. We assume $\sum_{k=1}^{K} P_k/K = 1$ and that P_k's follow some distribution as $K\to\infty$. A nominal posterior distribution is

$$p(\mathbf{d}|\mathbf{r},\mathbf{S};\sigma) = \frac{p(\mathbf{r}|\mathbf{d},\mathbf{S};\sigma)}{\sum_{\mathbf{e}} p(\mathbf{e})p(\mathbf{r}|\mathbf{e},\mathbf{S};\sigma)} \quad (2)$$

where σ is a control parameter in the place of the true noise variance. In Fig. 1 we introduce the *Bayes retrochannel* (BRC) which generates a stochastic estimate of the transmitted symbols by the distribution $p(\mathbf{d}|\mathbf{r},\mathbf{S};\sigma)$. A conditional mean estimator (CME) outputs the expectation of the stochastic output of the BRC, which is indeed $\mathsf{E}\{\mathbf{d}|\mathbf{r},\mathbf{S};\sigma\}$. In case $\sigma=\sigma_0$, the CME is equivalent to the individually optimal (IO) detector, while if $\sigma\to0$ it is the jointly optimal (JO) detector.

Interestingly, the BRC is equivalent to a spin glass in the sense that the distribution of its stochastic output is exactly the configuration distribution of the spin glass at thermal equilibrium. The performance of the optimal detector finds its counterpart as a certain macroscopic property of the spin glass, which can be solved using powerful tools in statistical mechanics in the large-system limit, i.e., when both K and N tend to infinity but with their ratio fixed as $K/N=\beta$.

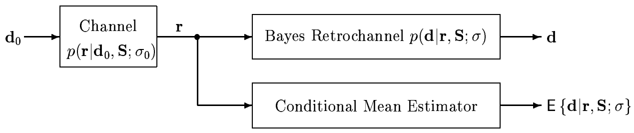

Figure 1: The conditional mean estimator.

The multiuser efficiency η, is

$$\eta = \frac{1}{1+\zeta\beta/\sigma_0^2} \quad (3)$$

where $\zeta = 1-2m+q$, and m and q satisfy

$$m = \mathsf{E}_{P,z}\left\{P\tanh\left(PE+\sqrt{PF}\,z\right)\right\} \quad (4a)$$
$$q = \mathsf{E}_{P,z}\left\{P\tanh^2\left(PE+\sqrt{PF}\,z\right)\right\} \quad (4b)$$
$$E = [\sigma^2+\beta(1-m)]^{-1} \quad (4c)$$
$$F = [\sigma^2+\beta(1-m)]^{-2}\cdot[\sigma_0^2+\beta(1-2m+q)] \quad (4d)$$

where $\mathsf{E}_{P,z}\{\cdot\}$ is taken over the received energy distribution of P and a standard Gaussian random variable z. It can be shown that m and q are the first and second moments of the CME output times the received energy conditioned on a transmitted symbol of $+1$. Consequently,

$$\zeta = \mathsf{E}\left\{P_k(d_k-d_{0k})^2\right\} \quad (5)$$

where k is a random user index, is the mean-square error of the CME output. In case of the IO detector, $F=E$ and $q=m$, so that the η is the solution to a fixed-point equation

$$\eta+\eta\frac{\beta}{\sigma_0^2}\left[1-\mathsf{E}_{P,z}\left\{P\tanh\left(\sqrt{\frac{P\eta}{\sigma_0^2}}\,z+\frac{P\eta}{\sigma_0^2}\right)\right\}\right]=1. \quad (6)$$

The BER of a user with energy P can be shown to be equal to $Q\left(\sqrt{\eta P}/\sigma_0\right)$. In the special case of equal power users, the formula obtained in [2] can be recovered by specializing the above results. Comparing (3) with the matched filter efficiency, which is also given by (3) by setting $\zeta=1$, we conclude that an optimum multiuser detector behaves as a matched filter with a user population reduced by a factor of ζ, or a spreading factor expanded by $1/\zeta$.

References

[1] D. Guo and S. Verdú, "Multiuser detection and statistical mechanics," to appear in *Ian Blake Festschrift*, 2002. http://www.princeton.edu/~dguo/research/.

[2] T. Tanaka, "A statistical mechanics approach to large-system analysis of CDMA multiuser detectors," *submitted to IEEE Trans. Inform. Theory*, 2002.

[1] This work was supported by the National Science Foundation under grant no. CCR0074277.