Decoupling of CDMA Multiuser Detection via the Replica Method

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Abstract

In CDMA, the optimal multiuser detector in mean square sense is the conditional mean estimator (CME) regardless of the input distribution, which outputs the expected value of the transmitted symbols conditioned on the received signal and the channel state (spreading sequences, fading, etc). In this paper, we first note that every multiuser detector can be reformulated as the CME for a certain postulated CDMA channel with a postulated input distribution which may be different from the actual channel and inputs. Using the replica method developed in statistical physics, a class of generalized CME front end applied to randomly spread CDMA is studied in a unified framework in the large-system limit. It is found that for any input distribution, the single-user channel seen at the generalized CME output for each user is equivalent to a Gaussian channel followed by a monotonic decision function. The degradation factor in the effective SNR of the equivalent channel due to multiple access interference is the multiuser efficiency, which is found to satisfy a fixed-point equation. The spectral efficiency of such a system under both joint and separate decoding are derived. Based on a general linear vector channel model, our results are also applicable to MIMO channels such as those arising in multiantenna systems.

1 Introduction

In code-division multiple access (CDMA), multiple-access interference (MAI) arises due to non-orthogonal spreading sequences from all users. Numerous multiuser detection techniques have been proposed to mitigate the MAI to various degrees. Regardless of the input distribution, the optimal detector in mean square sense is the *conditional mean estimator* (CME), which outputs the expected value of the input symbols conditioned on the received signal and the channel state (i.e., the spreading sequences, the received

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signal amplitudes, etc). In case of binary inputs, the CME is a soft version of the individually optimum detector which achieves the minimum bit-error-rate (BER) [1, 2]. A new standpoint we will establish in this paper is that every multiuser detector can be regarded as a CME for a certain postulated CDMA channel and input distribution, which may be different than the actual channel and inputs. In other words, a multiuser detector is nothing but a *generalized CME*, which is the optimal detector for a postulated CDMA system, and henceforth may be suboptimal for the actual system due to mismatch between the postulated and the actual probability laws.

The reformulation of multiuser detection as a generalized CME provides us with a unified framework in the treatment of multiuser communication problems. Indeed, by introducing an indefinite postulated channel and input distribution, it is viable to analyze a wide class of multiuser detectors and arrive at general conclusions. By tuning the postulated channel and inputs, the result can be easily particularized to most detectors of interest. Taking advantage of the above new formulation, we present in this paper a large-system characterization of the input-output relationship of a class of generalized CME front ends applied to randomly spread CDMA. By a large system we refer to the limit that both the number of users and the spreading factor tend to infinity with a fixed ratio.

It has been shown that the output of a wide class of linear multiuser detectors conditioned on the input converges in distribution to a Gaussian random variable in the large-system limit [3]. The reason is that the MAI in the detection output is a superposition of interferences and thus central limit theorems apply. Since the MAI seen by all users are statistically the same in the large-system limit regardless of the input distribution and the individual signal-to-noise ratios (SNR), the large-system performance of linear detectors can be fully characterized by a single parameter, called *multiuser efficiency*, which is the degradation in the effective SNR due to the MAI.[1] Fortunately, the multiuser efficiency of a finite size linear system can be written as an explicit function of the singular values of the spreading matrix, the empirical distributions of which converge to a known function as the matrix size goes to infinity. Therefore, the large-system multiuser efficiency can be obtained as an integral with respect to the limiting singular-value distribution, which, by using the Stieltjes transform, is found as the solution to a fixed-point equation (cf. e.g. [4, 3]). As far as linear multiuser detectors are concerned, the multiuser channel can be effectively decoupled into single-user Gaussian channels.

Much less success has been reported in the application of random matrix theory and central limit theorems in analyzing multiuser detectors that fall out of the above group of linear detectors, mainly due to lack of explicit expressions of the performance measures in terms of the singular values. For instance, the optimum detection output is notoriously non-Gaussian in the large-system limit. In this paper, we study the large-system input-output relationship of a class of generalized CME that corresponds to an arbitrary postulated input distribution and a postulated CDMA channel that differs from the actual channel only in the noise variance. Surprisingly, it is found that under arbitrary inputs and fading, the single-user channel seen at the generalized CME output is equivalent to a scalar Gaussian channel followed by a strictly monotonic decision function. Indeed, the fact that the output of nonlinear detectors conditioned on the input relates to a Gaussian distribution through a deterministic (linear or nonlinear) function has

[1] The concept of multiuser efficiency was first introduced in binary uncoded transmission to refer to the degradation of the minimum bit-error-rate (BER) relative to a single-user channel calibrated in the equivalent SNR [1].

long evaded discovery. It is also found in this paper that the effective SNR under this equivalent Gaussian channel is equal to the input SNR times the same multiuser efficiency for all users, which satisfies a fixed-point equilibrium equation. By appropriate choices of the postulated inputs and noise variance, the results can be particularized to obtain the multiuser efficiency of the matched filter, decorrelator, MMSE detector, as well as the jointly and individually optimum detectors. In all, the multiuser system under separate decoding can be effectively decoupled into single-user Gaussian channels that interact only through the multiuser efficiency.

The foundation of the above simple large-system characterization is the so-called "self-averaging" property, namely, the dependence of the performance measures on the spreading sequences vanishes as the system size increases without bound. This is a direct outcome of the asymptotic equipartition property (AEP). The "entropy rate" dictated by the AEP, also known as the free energy in statistical physics, is derived in this paper using the replica method. The replica method has its origin in spin glass theory in statistical physics, and was first used by Tanaka in multiuser detection to obtain the large-system uncoded minimum bit-error-rate and spectral efficiency with antipodal inputs [5]. The replica method has been used successfully in many problems in statistical physics as well as neural networks and coding theory, while a rigorous proof of the replica method is an ongoing effort in mathematics and physics communities.

Because of the decoupling shown in this paper, the capacity of the single-user channel seen at the generalized CME output is equal to the mutual information of the equivalent scalar Gaussian channel under the same inputs. The spectral efficiency under optimum joint decoding is also derived and it is found that regardless of the input distribution, successive decoding with a CME front end against the yet undecoded users achieves the optimum spectral efficiency.

From a practical viewpoint, this paper presents new results on the performance of CDMA under arbitrary signaling such as m-PAM. More importantly, the MAI, which often exhibits very complicated structure, is characterized by a single parameter, the multiuser efficiency. The spectral efficiency achievable by coded systems is also easily quantified by means of this parameter. Thus, our results offer valuable insights in the design and analysis of coded and uncoded CDMA systems.

The linear system in our study also models multiple-input multiple-output (MIMO) channels where the channel state is unknown at the transmitter. The results can thus be used to evaluate the performance of high-dimensional MIMO channels (such as multiple-antenna systems) with arbitrary signaling and various detection techniques.

2 CDMA and Multiuser Detection

Consider the K-user CDMA system with spreading factor N depicted in Fig. 1. At each interval the vector of input symbols from all users $\boldsymbol{X} = [X_1, \dots, X_K]^\top$ contains independent identically distributed (i.i.d.) entries with distribution p_X, which has zero mean and unit variance. The individual instantaneous SIRs $\{\Gamma_k\}_{k=1}^K$ are i.i.d. with distribution p_Γ of finite moments, hereafter referred to as the *SIR distribution*. Let the spreading sequence of user k be denoted by $\boldsymbol{s}_k = \frac{1}{\sqrt{N}}[S_{1k}, S_{2k}, \dots, S_{Nk}]^\top$, where S_{nk} are i.i.d. random variables with zero mean, unit variance, and finite moments. The $N \times K$ spreading matrix is denoted by $\boldsymbol{S} = [\sqrt{\Gamma_1}\,\boldsymbol{s}_1, \dots, \sqrt{\Gamma_K}\,\boldsymbol{s}_K]$. Assuming symbol-synchronous transmission,

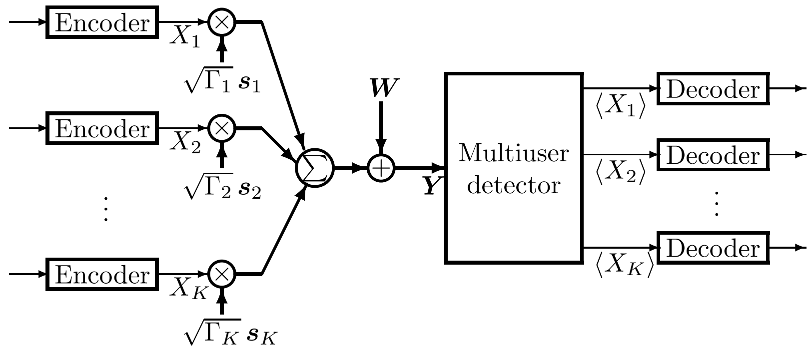

Figure 1: CDMA channel with separate decoding.

we have the following memoryless CDMA channel:

$$\boldsymbol{Y} = \boldsymbol{S}\boldsymbol{X} + \boldsymbol{W} \tag{1}$$

where $\boldsymbol{W} \sim \mathcal{N}(0, \boldsymbol{I})$. The characteristics of the channel is described as:

$$p_{\boldsymbol{Y}|\boldsymbol{X},\boldsymbol{S}}(\boldsymbol{y}|\boldsymbol{x}, \boldsymbol{S}) = (2\pi)^{-\frac{N}{2}} \exp\left[-\frac{1}{2}\|\boldsymbol{y} - \boldsymbol{S}\boldsymbol{x}\|^2\right]. \tag{2}$$

The most efficient use of channel (1) in terms of capacity is achieved by optimal joint decoding, where the total capacity subject to a certain input distribution is determined by the mutual information $I(\boldsymbol{X}; \boldsymbol{Y}|\boldsymbol{S})$. Practically, due to the prohibitive complexity of joint decoding, one often breaks the process into a multiuser detector front end followed by separate decoding as shown in Fig. 1. In this case, the CDMA channel together with the multiuser detector front end is viewed as a single-user channel for each user. The detection output sequence for an individual user is in general not a sufficient statistic for decoding even this user's own information; hence the loss in capacity.

One particular choice of the multiuser detector is the *conditional mean estimator*:

$$\langle \boldsymbol{X} \rangle \stackrel{\triangle}{=} \mathsf{E}\left\{\boldsymbol{X} \mid \boldsymbol{Y}, \boldsymbol{S}\right\}, \tag{3}$$

which achieves the minimum mean square error. Hereafter, angle brackets $\langle\cdot\rangle$ denote expectation with respect to the posterior probability distribution $p_{\boldsymbol{X}|\boldsymbol{Y},\boldsymbol{S}}$, which is determined by the input distribution p_X and the conditional Gaussian density function $p_{\boldsymbol{Y}|\boldsymbol{X},\boldsymbol{S}}$ of channel (1) through the Bayes formula.

The CME can be generalized to taking the conditional expectation as in (3) but with respect to the posterior probability distribution of a "postulated" CDMA system which may be different from the true one. Let the input distribution and channel characteristic of the postulated channel be $q_{\boldsymbol{X}}$ and $q_{\boldsymbol{Y}|\boldsymbol{X},\boldsymbol{S}}$ respectively, which, in turn, determines the postulated posterior probability distribution $q_{\boldsymbol{X}|\boldsymbol{Y},\boldsymbol{S}}$. The generalized CME output is conveniently denoted as

$$\langle \boldsymbol{X} \rangle_q \stackrel{\triangle}{=} \mathsf{E}_q\left\{\boldsymbol{X} \mid \boldsymbol{Y}, \boldsymbol{S}\right\}. \tag{4}$$

By choosing an appropriate probability measure q, it is possible to particularize the generalized CME to many different multiuser detectors of interest. In this paper, the postulated channel differs from the true channel (1) by only the noise variance:

$$q_{\boldsymbol{Y}|\boldsymbol{X},\boldsymbol{S}}(\boldsymbol{y}|\boldsymbol{x}, \boldsymbol{S}) = \left(2\pi\sigma^2\right)^{-\frac{N}{2}} \exp\left[-\frac{1}{2\sigma^2}\|\boldsymbol{y} - \boldsymbol{S}\boldsymbol{x}\|^2\right]. \tag{5}$$

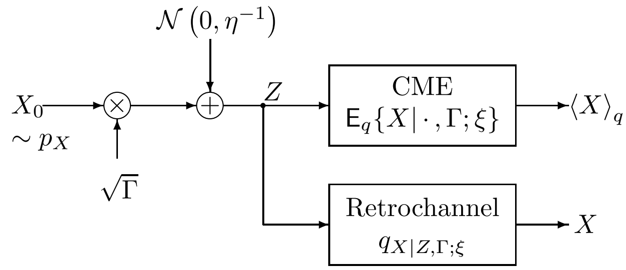

Figure 2: Equivalent scalar Gaussian channel, CME and retrochannel.

Here, σ serves as a control parameter. Also, the postulated inputs are i.i.d. with an arbitrary input distribution q_X of zero-mean and unit variance.

Suppose that the postulated input distribution q_X is standard Gaussian. It is not difficult to see that the generalized CME (4) outputs a linear filtering of the received signal:

$$\langle \boldsymbol{X} \rangle_q = \left[\boldsymbol{S}^\top \boldsymbol{S} + \sigma^2 \boldsymbol{I} \right]^{-1} \cdot \boldsymbol{S}^\top \cdot \boldsymbol{Y}. \tag{6}$$

If $\sigma \to \infty$, the generalized CME estimate is consistent with the matched filter output. If $\sigma = 1$, (6) is exactly the soft output of the linear MMSE detector. If $\sigma \to 0$, (6) converges to the output of the decorrelator.

Alternatively, suppose that the postulated input distribution q_X is identical to p_X. If $\sigma \to 0$, then $\lim_{\sigma\to 0} \langle \boldsymbol{X} \rangle_q$ is the jointly optimal (or maximum-likelihood) detection [2]. If $\sigma = 1$, then the postulated measure q coincides with the true measure p, and the CME outputs $\langle \boldsymbol{X} \rangle$ is a soft version of the individually optimum multiuser detector [2]. Also worth mentioning is that, if $\sigma \to \infty$, the generalized CME reduces to the matched filter.

3 Decoupling of CDMA Multiuser Detection

3.1 Main Results

Consider a canonical scalar Gaussian channel:

$$Z = \sqrt{\Gamma} \cdot X + \frac{1}{\sqrt{\eta}} \cdot W \tag{7}$$

where $\Gamma > 0$ is the input SIR, $\eta > 0$ the inverse noise variance and $W \sim \mathcal{N}(0,1)$. Given $\xi > 0$, we consider also a postulated Gaussian channel with input SIR Γ and inverse noise variance ξ. Let the input distribution to this postulated channel be q_X. Then the underlying measure of the postulated channel is $q_X \cdot q_{Z|X,\Gamma;\xi}$. A *retrochannel* of the postulated channel is characterized by the posterior probability distribution $q_{X|Z,\Gamma;\xi}$, namely, it takes in an input Z and outputs a random variable X according to $q_{X|Z,\Gamma;\xi}$. The generalized CME estimate of X given Z is therefore

$$\langle X \rangle_q \stackrel{\triangle}{=} \mathsf{E}_q \{ X \mid Z, \Gamma; \xi \}. \tag{8}$$

Consider now a concatenation of the scalar Gaussian channel (7) and the retrochannel of the postulated channel as depicted in Fig. 2. The generalized CME is also included. Let the input to the Gaussian channel (7) be denoted by X_0 to distinguish it from the retrochannel output X. We define the mean square error of the CME estimate and the variance of the retrochannel respectively:

$$\mathcal{E}(\Gamma;\eta,\xi) = \mathsf{E}\left\{ \left(X_0 - \langle X \rangle_q \right)^2 \middle| \Gamma;\eta,\xi \right\}, \tag{9}$$

$$\mathcal{V}(\Gamma;\eta,\xi) = \mathsf{E}\left\{ \left(X - \langle X \rangle_q \right)^2 \middle| \Gamma;\eta,\xi \right\}. \tag{10}$$

Claim 1 *Let the generalized CME of the CDMA channel (1) be defined by (4) with postulated input distribution q_X and noise variance σ^2. Then, in the large-system limit, the distribution of the multiuser detection output $\langle X_k \rangle_q$ conditioned on $X_k = x$ being transmitted with SIR Γ_k is identical to the distribution of the generalized estimate $\langle X \rangle_q$ of the equivalent scalar Gaussian channel (7) conditioned on $X = x$ being transmitted with input SIR $\Gamma = \Gamma_k$, where the multiuser efficiency η and the inverse noise variance ξ of the postulated scalar channel satisfy the coupled equations:*

$$\eta^{-1} = 1 + \beta\,\mathsf{E}\left\{\Gamma \cdot \mathcal{E}(\Gamma;\eta,\xi)\right\}, \tag{11a}$$

$$\xi^{-1} = \sigma^2 + \beta\,\mathsf{E}\left\{\Gamma \cdot \mathcal{V}(\Gamma;\eta,\xi)\right\}, \tag{11b}$$

where the expectations are taken over the SIR distribution p_Γ. In case of multiple solutions to (11), (η,ξ) are chosen to minimize the free energy:

$$\begin{aligned}\mathcal{F} = -\,\mathsf{E}\left\{ \int p_{Z|\Gamma;\eta}(z|\Gamma;\eta) \cdot \log q_{Z|\Gamma;\xi}(z|\Gamma;\xi)\,\mathrm{d}\,z \right\} - \frac{1}{2}\log\frac{2\pi}{\xi} - \frac{\xi}{2\eta}\log e \\ + \frac{\sigma^2\xi(\eta-\xi)}{2\beta\eta}\log e + \frac{1}{2\beta}\left[(\xi-1)\log e - \log\xi\right] + \frac{1}{2\beta}\log(2\pi) + \frac{\xi}{2\beta\eta}\log e.\end{aligned} \tag{12}$$

Claim 1 reveals that the multiple-access channel followed by a generalized CME can be decoupled into scalar Gaussian channels in the large-system limit, where the effect of the MAI is summarized as a single parameter η^{-1} representing the noise enhancement. Note that conditioned on the input, the multiuser CME output is not Gaussian; rather, it is asymptotically a function (the generalized CME is nothing but a decision function) of a Gaussian random variable. It is in general not difficult to find solutions to (11) numerically. Multiple solutions may coexist, which is known as the phenomenon of phase transitions in statistical physics. Among those solutions, the multiuser efficiency is the one that gives the smallest value of the free energy (12), which carries relevant operational meaning in the communication problem.

The decision function (i.e., the CME) in Fig. 2 is strictly increasing and thus inconsequential in both detection and information theoretic viewpoints. Hence the following:

Corollary 1 *In the large-system limit, the single-user channel capacity at the generalized CME output is equal to the input-output mutual information of the equivalent scalar Gaussian channel (7) with the same input distribution and SIR, and an inverse noise variance η as the multiuser efficiency determined by Claim 1:*

$$\mathsf{C}(\Gamma;\eta) = \mathsf{D}(p_{Z|X,\Gamma;\eta} \,||\, p_{Z|\Gamma;\eta} \,|\, p_X). \tag{13}$$

Clearly, the overall spectral efficiency under suboptimal separate decoding is

$$\mathsf{C}_{\mathrm{sep}}(\beta) = \beta\,\mathsf{E}\left\{\mathsf{C}(\Gamma;\eta)\right\}. \tag{14}$$

The optimal spectral efficiency under joint decoding is greater than (14). The optimal joint decoding gain is given by the following:

Claim 2 *The gain of optimal joint decoding over the multiuser CME followed by separate decoding in the large-system spectral efficiency of the CDMA channel (1) is*[2]

$$\mathsf{C}_{\text{joint}}(\beta) - \mathsf{C}_{\text{sep}}(\beta) = \frac{1}{2}[(\eta - 1)\log e - \log \eta] = \mathsf{D}\left(\mathcal{N}(0,\eta)\,||\,\mathcal{N}(0,1)\right), \tag{15}$$

where η is the CME multiuser efficiency.

The expression (15) coincides with the expression found originally in [6] in the case of Gaussian inputs and later in [7] in the case of binary inputs. Interestingly, the spectral efficiencies under joint and separate decoding are also related by the following generalization of a result in [8]:

Proposition 1 *Under every input distribution p_X,*

$$\mathsf{C}_{\text{joint}}(\beta) = \int_0^\beta \frac{1}{\beta'}\mathsf{C}_{\text{sep}}(\beta')\,\mathrm{d}\beta'. \tag{16}$$

3.2 Discussions

We can conceive an interference canceler that decodes the users successively in which reliably decoded symbols are used to reconstruct the interference for cancellation. Suppose the users are decoded in reverse order, then the generalized CME for user k sees only $k-1$ interfering users. Hence the performance for user k under successive decoding is identical to that of the CME applied to a CDMA system with k instead of K users. The multiuser efficiency experienced by user k is $\eta\left(\frac{k}{N}\right)$ where use the fact that it is a function of the load $\frac{k}{N}$ seen by the generalized CME for user k. It is easy to see that the overall spectral efficiency converges almost surely:

$$\frac{1}{N}\sum_{k=1}^{K}\mathsf{C}\left(\Gamma_k;\eta\left(\frac{k}{N}\right)\right) \to \mathsf{E}\left\{\int_0^\beta \mathsf{C}(\Gamma;\beta')\,\mathrm{d}\beta'\right\}. \tag{17}$$

Note that the above result on successive decoding is true for arbitrary input distribution and generalized CME detectors. In the special case of the CME, for which the postulated inputs and channel are identical to the actual input and channel, the right hand side of (17) is equal to $\mathsf{C}_{\text{joint}}(\beta)$ by Proposition 1. We can summarize this principle as:

Proposition 2 *In the large-system limit, successive decoding with a CME front end against yet undecoded users achieves the optimal CDMA channel capacity under arbitrary input distributions.*

Proposition 2 is a generalization of the previous result that a successive canceler with a linear MMSE front end against undecoded users achieves the capacity of the CDMA channel under Gaussian inputs (cf. e.g. [9]). Indeed, this result is an outcome of the chain rule of mutual information, which holds for all inputs and arbitrary number of users.

3.3 Analysis via the Replica Method

The statistical inference problem faced by the decoder is depicted in Fig. 3. The input and output of the channel $p_{\boldsymbol{Y}|\boldsymbol{X},\boldsymbol{S}}$ under state $\boldsymbol{S}$ is denoted by $\boldsymbol{X}_0$ and $\boldsymbol{Y}$ respectively. A

[2] The base of logarithm is indefinite and agrees with the unit of information measure.

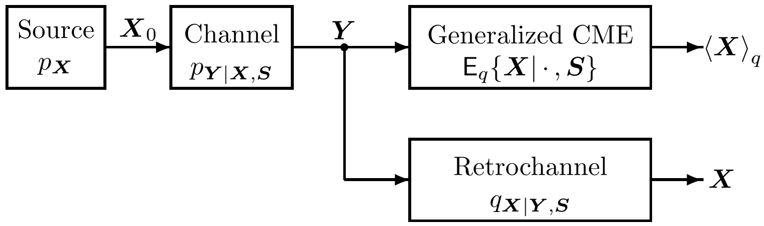

Figure 3: Canonical channel, retrochannel and generalized CME.

generic detector regards $\boldsymbol{Y}$ as the output of a postulated channel $q_{\boldsymbol{Y}|\boldsymbol{X},\boldsymbol{S}}$ caused by some input $\boldsymbol{X}$ with distribution $q_{\boldsymbol{X}}$. One meaningful way of processing the received signal is to find the expected value of $\boldsymbol{X}$ conditioned on $(\boldsymbol{Y},\boldsymbol{S})$, i.e., $\langle\boldsymbol{X}\rangle_q = \mathsf{E}_q\{\boldsymbol{X}|\boldsymbol{Y},\boldsymbol{S}\}$. A postulated measure q different than p may be either due to lack of knowledge of the true statistics or a particular choice that corresponds to a certain detector of interest. The retrochannel induced by the postulated channel, upon receiving $\boldsymbol{Y}$ under channel state $\boldsymbol{S}$, outputs a random variable $\boldsymbol{X}$ according to $q_{\boldsymbol{X}|\boldsymbol{Y},\boldsymbol{S}}$. Clearly, the generalized CME output $\langle\boldsymbol{X}\rangle_q$ is the expected value of the retrochannel output $\boldsymbol{X}$ given $(\boldsymbol{Y},\boldsymbol{S})$.

This paper studies the distribution of the detection output $\langle X_k\rangle_q$ conditioned on the input X_{0k} in the large-system limit, where both the number of users K and the spreading factor N tend to infinity but with K/N converging to a positive number β. Here, we use X_{0k} to denote the input to distinguish it from the retrochannel output X_k. Although the input-output relationship is dependent on the spreading matrix for any finite size system, it becomes increasingly predictable from merely a few parameters independent of the realization of the spreading matrix as the system size grows without bound. This is known as the *self-averaging property* in statistical physics. It is a direct outcome of the asymptotic equipartition property. In the CDMA context, the AEP ensures a strong consequence that for almost all realizations of the data, the received signal and the spreading sequences, macroscopic quantities such as the BER, the output SIR and the spectral efficiency, averaged over data, converge to deterministic quantities in the large-system limit.

In order to reveal the input-output relationship, we calculate the joint moments conditioned on the channel state:

$$\mathsf{E}\left\{ X_{0k}^j \cdot \langle X_k\rangle_q^i \,\middle|\, \boldsymbol{S}\right\}, \quad i,j = 0,1,\ldots \tag{18}$$

and then infer the distribution of $(\langle X_k\rangle_q - X_{0k})$. It is helpful to introduce independent replicas of the retrochannel output $\boldsymbol{X}$, denoted as $\boldsymbol{X}_1,\ldots,\boldsymbol{X}_u$. Since $\boldsymbol{X}_0 \to (\boldsymbol{Y},\boldsymbol{S}) \to (\boldsymbol{X}_1,\ldots,\boldsymbol{X}_u)$ is a Markov chain, it can be shown that (18) is equivalent to the joint moments of the head and tail of the Markov chain

$$\mathsf{E}\left\{ X_{0k}^j \cdot \prod_{m=1}^{i} X_{mk} \,\middle|\, \boldsymbol{S}\right\}, \quad i,j = 0,1,\ldots \tag{19}$$

which can be evaluated by integrating over the distribution of the central random variable $\boldsymbol{Y}$ conditioned on $\boldsymbol{S}$.

According to the AEP, the randomness in the following quantity:

$$-\frac{1}{K}\log q_{\boldsymbol{Y}|\boldsymbol{S}}(\boldsymbol{Y}|\boldsymbol{S}) \tag{20}$$

vanishes as $K \to \infty$. The limit is known as the *free energy* in statistical physics, which is denoted by:

$$\mathcal{F} \triangleq -\lim_{K\to\infty} \mathsf{E}\left\{\frac{1}{K}\log q_{\boldsymbol{Y}|\boldsymbol{S}}(\boldsymbol{Y}|\boldsymbol{S})\right\}. \tag{21}$$

Essentially, in the large-system limit, almost all realizations of the received signal are "typical", and it suffices to calculate the moments in the typical set. The expected value of the logarithm in (21) is an open problem, which can be formulated equivalently as

$$\mathcal{F} = -\lim_{K\to\infty}\frac{1}{K}\lim_{u\to 0}\frac{\partial}{\partial u}\log \mathsf{E}\left\{q^u_{\boldsymbol{Y}|\boldsymbol{S}}(\boldsymbol{Y}|\boldsymbol{S})\right\}. \tag{22}$$

Using the replicas of the retrochannel output, we can evaluate

$$-\lim_{K\to\infty}\frac{1}{K}\log \mathsf{E}\left\{q^u_{\boldsymbol{Y}|\boldsymbol{S}}(\boldsymbol{Y}|\boldsymbol{S})\right\} = -\lim_{K\to\infty}\frac{1}{K}\log \mathsf{E}\left\{\prod_{a=1}^{u} q_{\boldsymbol{Y}|\boldsymbol{X},\boldsymbol{S}}(\boldsymbol{y}|\boldsymbol{X}_a,\boldsymbol{S})\right\} \tag{23}$$

as a function of the integer replica number u. Ubiquitous in statistical physics, the *replica trick* assumes that the order of the limit and the derivative in (22) can be exchanged, and that the resulting expression from (23) is also valid at least in the vicinity of $u = 0$, and therefore finds the derivative at $u = 0$ as the free energy. Note that $(\boldsymbol{Y}, \boldsymbol{S})$ is induced by the transmitted symbols $\boldsymbol{X}_0$. By taking expectation over $\boldsymbol{Y}$ first and then averaging over the $\boldsymbol{S}$, one finds that

$$\frac{1}{K}\log \mathsf{E}\left\{q^u_{\boldsymbol{Y}|\boldsymbol{S}}(\boldsymbol{Y}|\boldsymbol{S})\right\} = \frac{1}{K}\log \mathsf{E}\left\{\exp\left[\beta^{-1}K\cdot G_K^{(u)}\left(\boldsymbol{\Gamma},[\boldsymbol{X}_0,\dots,\boldsymbol{X}_u]\right)\right]\right\} \tag{24}$$

where $G_K^{(u)}$ is some function of the SIRs, the transmitted symbols and their replicas. By first conditioning on the $(u+1)\times(u+1)$ correlation matrix $\boldsymbol{Q}$ of the replicas, the central limit theorem helps to reduce (24) to

$$\frac{1}{K}\log\int \exp\left[\beta^{-1}K\cdot G^{(u)}(\boldsymbol{Q})\right]\mu_K^{(u)}(\,\mathrm{d}\boldsymbol{Q}) \tag{25}$$

where $G^{(u)}$ is some function of the correlation matrix $\boldsymbol{Q}$, and $\mu_K^{(u)}$ is its probability measure. Large deviations can be invoked to show that (25) converges as $K \to \infty$ to

$$\sup_{\boldsymbol{Q}}[\beta^{-1}\cdot G^{(u)}(\boldsymbol{Q}) - I^{(u)}(\boldsymbol{Q})] \tag{26}$$

where $I^{(u)}$ is the rate function of the measure $\mu_K^{(u)}$ [10]. Seeking the extremum (26) over a $(u+1)^2$-dimensional space is a hard problem. The technique to circumvent this is to assume *replica symmetry*, namely, that the supremum in $\boldsymbol{Q}$ is symmetric over all replicated dimensions. The resulting supremum is then over a few parameters, and the free energy can be obtained.

The joint moments (19) of the input and the retrochannel output can be evaluated for all "typical" realizations of the received signal. The result is that $\langle X_k\rangle_q$ can be regarded as the generalized conditional mean estimate of a scalar Gaussian channel with input X_{0k}, hence the proof of Claim 1. Given the input distribution p_X, the total capacity under optimum joint decoding converges to

$$\mathsf{C}_{\text{joint}}(\beta) = \beta\mathcal{F}|_{q=p} - \frac{1}{2}\log(2\pi e). \tag{27}$$

Claim 2 is proved by finding the free energy under the assumption that the postulated measure q is identical to the actual measure p.

4 Conclusion

Using the replica method, a family of generalized conditional mean estimators is studied in the large-system limit, which includes well-known detectors such as the matched filter, decorrelator, MMSE detector, the jointly and individually optimum detector. One major result is the decoupling of Gaussian CDMA channels concatenated with a multiuser detector front end into scalar Gaussian channels. Another result is general formulas for the spectral efficiency of CDMA channels expressed in terms of the multiuser efficiency. It is straightforward to particularize the results to any practical input constellation, such as m-QAM, which can be useful not only for the design and analysis of CDMA, but for the important special case of the canonical single-user multiantenna array channel.

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