# Spectral Efficiency of Large-system CDMA via Statistical Physics

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*Abstract* — **The channel capacity of randomly spread CDMA subject to Gaussian noise and flat fading is studied in the large-system limit under arbitrary input distributions. Using the replica method originally developed in statistical physics, we find the spectral efficiencies under both joint decoding and single-user decoding. We show that under single-user decoding, where a conditional mean estimator is first applied to separate the users, each user is as if transmitting through a degraded Gaussian channel where the ratio of the effective energy to the true energy is the same multiuser efficiency to everyone. It is found that the multiuser efficiency is the solution to a fixed-point equilibrium equation. The spectral efficiency under both joint and separate decoding are expressed explicitly in the multiuser efficiency, the input distribution and the fading characteristics.**

## I Introduction

In code-division multiple access (CDMA), unless the spreading sequences of all users are mutually orthogonal, so that single-user capacity is enjoyed by everyone, there is some inherent loss in spectral efficiency due to the imposed spreading structure. The loss was quantified under random spreading in the large-system limit using random matrix theory by Grant and Alexander [1] and Verdú and Shamai [2]. The complexity of optimal joint decoding, necessary for achieving the capacity, is often prohibitive in practice. A common strategy is to separate the users using a multiuser detector front end and then perform single-user decoding, which induces further penalty on spectral efficiency. Assuming Gaussian inputs, the penalty was found for linear detectors as a function of the multiuser efficiency of the linear front end in [2], also using random matrix theory.

The success of random matrix theory in the analysis of the spectral efficiency hinges on the fact that the performance measures for a finite number of users ($K$) and spreading factor ($N$) can be written as explicit functions of the eigenvalues of the spreading matrix [3], the empirical distributions of which converge to a known function in the large-system limit where both $K$ and $N$ tend to infinity but with their ratio fixed [4]. As an important consequence, the dependency of the performance measures of randomly spread CDMA on the spreading sequences diminishes with probability 1 in the large-system limit. In other words, the performance measures are self-averaging. Random matrix theory underlies many interesting asymptotic results. In particular, the multiuser efficiency of the MMSE detector is found as a solution to the Tse-Hanly fixed-point equation [5]. The spectral efficiency of Gaussian CDMA channels subject to flat fading was found by Shamai and Verdú [6].

All the above results on spectral efficiency assume Gaussian inputs which are optimum because the channel characteristics are known at the receiver. Often used in practice are signal constellations that are more friendly to information bits, such as QPSK, 16QAM, etc. Therefore, it is of great interest to analyze the effects of given input distributions. Random matrix theory is found to be quite limited in this regard.

The self-averaging property is nothing but a manifestation of a fundamental law of nature that the fluctuation of macroscopic properties of certain many-body systems vanishes in thermodynamic limit when the number of interacting bodies becomes sufficiently large. In CDMA, the self-averaging principle ensures a strong property that for almost all realizations of the noise process and the spreading sequences, certain macroscopic average over the posterior probability distribution converges to the same number in the large-system limit, which is its ensemble average over the distribution of the noise and the spreading sequences. In [7], Tanaka introduced statistical physics concepts and methodologies into CDMA and obtained the bit-error-rate and the spectral efficiency of Gaussian CDMA under binary input constraint. Guo and Verdú further elucidated the relationship between statistical physics and CDMA and generalized Tanaka's results to the case of unbalanced received powers [8]. Inspired by [7], Müller and Gerstacker studied the channel capacity under separated decoding, where an optimal multiuser detector with no knowledge of the error-control codes is applied to generate soft decision for each user for single-user decoding [9]. They noticed that the capacity loss due to separation of detection and decoding under binary input constraint takes the same formula as the capacity loss under Gaussian distributed inputs. Müller further conjectured that the capacity loss is given by the same expression regardless of the input distribution [10].

In this paper, we present some new results on Gaussian CDMA spectral efficiency under arbitrary input distribution and fading characteristics. Analytical expressions are obtained through replica analysis, a powerful tool originally developed in spin glass theory in statistical physics. Under separate decoding, assuming that a detector outputs the mean value of the input symbol of each user conditioned on the received signal and the spreading sequences for subsequent independent decoding, we find that each resulting single-user channel is equivalent to a degraded Gaussian channel. The ratio of the effective energy to the true energy in the degraded Gaussian channel, called the *multiuser efficiency*, is found to satisfy a fixed-point equilibrium equation (a generalization of the Tse-Hanly equation [5]), and is the same for all users. The spectral efficiencies, both under joint and separate decoding, are found in explicit expressions in the multiuser efficiency, the input distribution and the received energy distribution. As an immediate corollary, Müller's conjecture on the capacity loss is true, and we identify the loss as a Kullback-Leibler

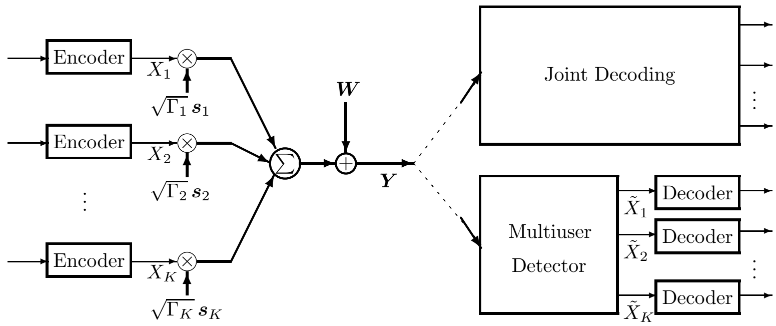


Figure 1: System model of CDMA with joint or separate decoding.

divergence between two Gaussian distributions.

The linear system in our study also models multiple-input multiple-output (MIMO) channels where the channel state is unknown at the transmitter. Our results are equally applicable to evaluating the spectral efficiency of MIMO channels with high dimensionality under constellation constraints. An example is multiple antenna systems under homogeneous fading.

## II System Model

We study a $K$-user CDMA system with a spreading factor of $N$ depicted in Fig. 1. We assume that all users employ the same type of signaling. All symbols are independent identically distributed (i.i.d.) with distribution $p_X$, normalized so that $\mathsf{E}\left\{X^2\right\} = 1$. We use $\boldsymbol{X} = [X_1, \dots, X_K]^\top$ to denote a vector of input symbols from the $K$ users. Throughout this paper, random variables are denoted by upper case letters. An expectation $\mathsf{E}\{\cdot\}$ is taken over the joint distribution of the random variables within the braces.

Let user $k$'s spreading sequence be denoted by $\boldsymbol{s}_k = \frac{1}{\sqrt{N}}[s_{1k}, s_{2k}, \dots, s_{Nk}]^\top$, and the $N \times K$ spreading matrix denoted by $\boldsymbol{S} = [\boldsymbol{s}_1, \dots, \boldsymbol{s}_K]$, where the $s_{nk}$'s are i.i.d. random variables with zero mean, unit variance and finite higher-order moments. Let $\Gamma_1, \dots, \Gamma_K$ be the $K$ users' respective received energies per symbol and $\boldsymbol{A} = \mathrm{diag}(\sqrt{\Gamma_1}, \dots, \sqrt{\Gamma_K})$. Assuming symbol-synchronous transmission, we have the following memoryless multiple-access channel:

$$\boldsymbol{Y} = \boldsymbol{SAX} + \boldsymbol{W} \tag{1}$$

where $\boldsymbol{W}$ is a vector that contains i.i.d. zero-mean Gaussian random variables with unit variance. The characteristic of the Gaussian CDMA channel with flat fading can be described as

$$p_{\boldsymbol{Y}|\boldsymbol{X},\boldsymbol{S}}(\boldsymbol{y}|\boldsymbol{x},\boldsymbol{S}) = (2\pi)^{-\frac{N}{2}} \exp\left[-\frac{1}{2}\|\boldsymbol{y} - \boldsymbol{SAx}\|^2\right] \tag{2}$$

where $\|\cdot\|$ denotes the Euclidean norm of a vector. Note that the spreading sequences are randomly chosen for each user and not dependent on the received energies. The channel (1) is the real-valued counterpart of the more general complex fading channel model (e.g., in [6]).

The input signal-to-interference ratio (SIR) is defined as the energy ratio of the useful signal to the noise in the matched filter output in absence of interfering users, which is conveniently $\Gamma_k$ for user $k$ in our setting. We assume that the SIR of all users are known deterministic numbers, and as $K \to \infty$, their empirical cumulative distributions converge to a known distribution $P_\Gamma$, hereafter referred to as the *SIR distribution*. The SIR distribution captures the overall effect of the transmit energies, the noise level, and the channel fading characteristics.

The total capacity of the CDMA channel subject to a certain input distribution is the mutual information between the transmitted symbols $\boldsymbol{X}$ and the received signal $\boldsymbol{Y}$. There exists an error-correcting code of any rate no larger than the capacity such that by jointly decoding all users as depicted by the upper right block in Fig. 1, the original information can be recovered arbitrarily reliably.

Practically, we often break the process into multiuser detection followed by separate decoding, as shown by the lower right blocks in Fig. 1. A multiuser detector front end outputs an estimate of the transmitted symbols without knowledge of the error-control codes used by the encoder. Each decoder takes only the decision statistic for a single user of interest for decoding without awareness of the existence of any other users. We can regard the CDMA channel and the multiuser detector together as a superchannel, which is equivalent to $K$ separate single-user channels. By the data processing theorem, the capacity of the superchannel, which is the sum of the single-user channel capacities, is less than the capacity of the original CDMA channel. One has to restrict the power of the multiuser detector here; otherwise the detector could in principle encode the received signal vector and the spreading sequences into a single real number as its output, which is a sufficient statistic for all users! In this paper, we study a particular type of detector, namely, the conditional mean estimator, which outputs the mean value of the symbols conditioned on the received signal and the spreading sequences.

Clearly, the spectral efficiency in either joint or single-user (separate) decoding is dependent on the spreading sequences. In this paper, we are concerned with only the large-system regime, namely, when both $K$ and $N$ tend to infinity but with $K/N$, known as the system load, converging to a fixed positive number $\beta$. In such asymptote, every performance measure we are interested in converges for almost all choices of the spreading sequences. Consequently, we can describe the multiuser efficiency and the spectral efficiency of the channel and the superchannel using merely the macroscopic parameters without worrying about the instantaneous spreading sequences.

## III CDMA and the Replica Method

In CDMA, the posterior distribution can be obtained from the prior distribution and the conditional distribution through the Bayes formula

$$\begin{aligned} & p_{\boldsymbol{X}|\boldsymbol{Y},\boldsymbol{S}}(\boldsymbol{x}|\boldsymbol{y},\boldsymbol{S}) \\ = & \ \frac{p_{\boldsymbol{Y}|\boldsymbol{X},\boldsymbol{S}}(\boldsymbol{y}|\boldsymbol{x},\boldsymbol{S}) \cdot p_{\boldsymbol{X}}(\boldsymbol{x})}{p_{\boldsymbol{Y}|\boldsymbol{S}}(\boldsymbol{y}|\boldsymbol{S})} \quad (3) \\ = & \ Z^{-1}(\boldsymbol{y},\boldsymbol{S}) \cdot p_{\boldsymbol{X}}(\boldsymbol{x}) \cdot \exp\left[-\frac{1}{2}\|\boldsymbol{y}-\boldsymbol{S}\boldsymbol{A}\boldsymbol{x}\|^2\right] \quad (4) \end{aligned}$$

where in (3), $p_{\boldsymbol{Y}|\boldsymbol{S}}(\boldsymbol{y}|\boldsymbol{S})$ is the marginal distribution of $p_{\boldsymbol{Y},\boldsymbol{X}|\boldsymbol{S}}(\boldsymbol{y},\boldsymbol{x}|\boldsymbol{S}) = p_{\boldsymbol{Y}|\boldsymbol{X},\boldsymbol{S}}(\boldsymbol{y}|\boldsymbol{x},\boldsymbol{S})p_{\boldsymbol{X}}(\boldsymbol{x})$, and in (4),

$$\begin{aligned} Z(\boldsymbol{y},\boldsymbol{S}) &= (2\pi)^{\frac{N}{2}} \cdot p_{\boldsymbol{Y}|\boldsymbol{S}}(\boldsymbol{y}|\boldsymbol{S}) \quad (5) \\ &= \mathsf{E}\left\{ \exp\left[-\frac{1}{2}\|\boldsymbol{y}-\boldsymbol{S}\boldsymbol{A}\boldsymbol{X}\|^2\right] \,\middle|\, \boldsymbol{S} \right\} \quad (6) \end{aligned}$$

is a normalizing coefficient, which we conveniently refer to as the *partition function*. Note that the expectation in (6) is taken over $\boldsymbol{X}$ conditioned on $\boldsymbol{S}$.

### A Spectral Efficiency of Joint Decoding

Since the input distribution is fixed, the total capacity under joint decoding is equal to the input-output mutual information conditioned on the spreading matrix

$$I(\boldsymbol{X};\boldsymbol{Y}|\boldsymbol{S}) = \mathsf{E}\left\{ \log \frac{p_{\boldsymbol{Y}|\boldsymbol{X},\boldsymbol{S}}(\boldsymbol{Y}|\boldsymbol{X},\boldsymbol{S})}{p_{\boldsymbol{Y}|\boldsymbol{S}}(\boldsymbol{Y}|\boldsymbol{S})} \,\middle|\, \boldsymbol{S} \right\} \quad (7)$$

where the expectation is taken over the joint conditional distribution $p_{\boldsymbol{Y},\boldsymbol{X}|\boldsymbol{S}}$. Noticing that the channel characteristics given by (2) is a Gaussian density, we have easily

$$\mathsf{E}\left\{ \log p_{\boldsymbol{Y}|\boldsymbol{X},\boldsymbol{S}}(\boldsymbol{Y}|\boldsymbol{X},\boldsymbol{S}) \,\middle|\, \boldsymbol{S} \right\} = -\frac{N}{2}\log(2\pi e). \quad (8)$$

The spectral efficiency is defined as the total capacity divided by the number of chips per symbol interval. Therefore, by (5), (7) and (8), the spectral efficiency of joint decoding is

$$\mathsf{C}(\boldsymbol{S}) = \frac{1}{N} I(\boldsymbol{X};\boldsymbol{Y}|\boldsymbol{S}) = -\frac{1}{N}\cdot \mathsf{E}\left\{ \log Z(\boldsymbol{Y},\boldsymbol{S}) | \boldsymbol{S} \right\} - \frac{1}{2}. \quad (9)$$

Interestingly, the spectral efficiency is closely related to the partition function $Z(\boldsymbol{Y},\boldsymbol{S})$.

We define the *free energy* as

$$-\frac{1}{K}\log Z(\boldsymbol{Y},\boldsymbol{S}). \quad (10)$$

It includes all information about statistics of the observables in the system. As a macroscopic property, the free energy converges with probability 1 to its expectation over the distribution of the random variables $(\boldsymbol{Y},\boldsymbol{S})$ in the large-system limit, denoted by $\mathcal{F}$,

$$\mathcal{F} = -\lim_{K\to\infty} \mathsf{E}\left\{ \frac{1}{K}\log Z(\boldsymbol{Y},\boldsymbol{S}) \right\}. \quad (11)$$

Therefore, the spectral efficiency under joint decoding converges almost surely to

$$\mathsf{C} = \beta\mathcal{F} - \frac{1}{2} \quad (12)$$

in the large-system limit. Note that the $-1/2$ term in (12) can be removed by redefining the partition function up to a constant coefficient. In either way, the spectral efficiency is affine in the free energy.

### B Spectral Efficiency of Separate Decoding

A conditional mean estimator is used to separate the users and outputs

$$\tilde{\boldsymbol{X}} = \langle \boldsymbol{X} \rangle \stackrel{\triangle}{=} \mathsf{E}\left\{ \boldsymbol{X} \mid \boldsymbol{Y}, \boldsymbol{S} \right\} \quad (13)$$

where, by definition, the operator $\langle\cdot\rangle$ gives the expectation taken over the posterior probability distribution $p_{\boldsymbol{X}|\boldsymbol{Y},\boldsymbol{S}}$. The reason for choosing this particular detector is two-fold: 1) This detector is optimal in the sense that it has the minimum output mean square error; and 2) it bridges the spectral efficiencies of joint and separate decoding, as we will see later.

The single-user channel capacity due to the multiuser detector for an arbitrary user $k$ is equal to the mutual information between the input symbol and the detector output

$$I(X_k; \langle X_k \rangle \mid \boldsymbol{S}). \quad (14)$$

For the purpose of calculating (14), we study the distribution of $\langle X_k \rangle$ conditioned on the input $X_k$. Our approach is to calculate joint moments

$$\mathsf{E}\left\{ X_k^j \cdot \langle X_k \rangle^i \,\middle|\, \boldsymbol{S} \right\}, \quad i,j = 0,1,\ldots \quad (15)$$

and then infer the distribution of $(\langle X_k \rangle - X_k)$. It turns out that the moments are macroscopic properties, which converge with probability 1 in the large-system limit. It therefore suffices to calculate

$$\lim_{K\to\infty} \mathsf{E}\left\{ X_k^j \cdot \langle X_k \rangle^i \right\}. \quad (16)$$

It is conceptually helpful here to introduce a stochastic estimator called the *Bayes retrochannel* [8]. Upon a received signal $\boldsymbol{Y}$ with a channel state $\boldsymbol{S}$, this stochastic estimator outputs a random variable according to the probability distribution $p_{\boldsymbol{X}|\boldsymbol{Y},\boldsymbol{S}}$ given by (3). It is clear that the expected value of the retrochannel output is exactly the conditional mean estimate. To distinguish the original symbols that caused the received signal from the retrochannel output, we denote the original symbol vector by $\boldsymbol{X}_0$, and the retrochannel output by $\boldsymbol{X}$. We note that $\boldsymbol{X}_0 \to (\boldsymbol{Y},\boldsymbol{S}) \to \boldsymbol{X}$ is a Markov chain. It can be shown that (16) is tantamount to

$$\lim_{K\to\infty} \mathsf{E}\left\{ X_{0k}^j \cdot X_k^i \right\}, \quad (17)$$

which turns out to be easier to calculate.

We have distilled the spectral efficiency problem under both joint and separate decoding to calculating some ensemble averages, namely, the free energy (11) and the moments (17). We resort to the replica method, a powerful technique developed in statistical physics, to calculate these quantities.

### C Replica Method

The difficulty of calculating the free energy (11) is due to the unwieldiness of taking the average of a logarithm. An alternative is to write

$$\begin{aligned} \mathcal{F} &= -\lim_{K\to\infty}\frac{1}{K}\lim_{u\to 0}\frac{\partial}{\partial u}\log \mathsf{E}\left\{ Z^u(\boldsymbol{Y},\boldsymbol{S}) \right\} \quad (18) \\ &= -\lim_{u\to 0}\frac{\partial}{\partial u}\lim_{K\to\infty}\frac{1}{K}\log \mathsf{E}\left\{ Z^u(\boldsymbol{Y},\boldsymbol{S}) \right\} \quad (19) \end{aligned}$$

where (18) can be easily verified and (19) is resulted by assuming that the order of the limit in $K$ and the limit of the derivative in $u$ can be exchanged. For an arbitrary integer $u$,

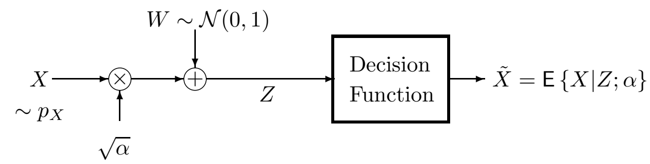


Figure 2: Equivalent single-user channel.

we introduce $u$ replicas of the CDMA system with the same received signal $\boldsymbol{Y}$ and spreading matrix $\boldsymbol{S}$. The partition function of the replicated system is

$$Z^u(\boldsymbol{y}, \boldsymbol{S}) = \mathsf{E}\left\{ \prod_{a=1}^{u} \exp\left[ -\frac{1}{2} \|\boldsymbol{y} - \boldsymbol{S}\boldsymbol{A}\boldsymbol{X}_a\|^2 \right] \middle| \, \boldsymbol{S} \right\} \quad (20)$$

where the expectation is taken over the the i.i.d. replicated symbols, $\{X_{ak} | a = 1, \ldots, u,\, k = 1, \ldots, K\}$, conditioned on $\boldsymbol{S}$. We can henceforth evaluate

$$-\lim_{K\to\infty} \frac{1}{K} \log \mathsf{E}\{Z^u(\boldsymbol{Y}, \boldsymbol{S})\} \quad (21)$$

as a function of the integer $u$. We then use a trick: We assume that the resulting expression is also valid for an arbitrary real number $u$ and find the derivative at $u = 0$ as the free energy. This is called the *replica method*. The replica method was invented in the context of spin glasses [11] and has since been successfully applied to many problems [12]. There are intensive ongoing efforts in the mathematics and physics community to find a rigorous proof for the replica method which we shall avoid in this work.

The replica idea is also used to calculate (17). We use $\boldsymbol{X}_0$ to denote the transmitted symbols and use $\{\boldsymbol{X}_a\}$ to represent the replicated symbols. This can be best understood by considering $u$ replicas of the Bayes retrochannel with the same received signal $\boldsymbol{Y}$ and spreading matrix $\boldsymbol{S}$. Clearly, $\boldsymbol{X}_0 \to (\boldsymbol{Y}, \boldsymbol{S}) \to \{\boldsymbol{X}_a\}$ is a Markov chain. The moments (17) are tantamount to

$$\lim_{K\to\infty} \mathsf{E}\left\{ X_{0k}^j \cdot \prod_{m=1}^{i} X_{mk} \right\} \quad (22)$$

which can be evaluated by working with a modified partition function than (20).

## IV New Results

We present our new results without proofs. Readers interested in details are referred to [13]. Assuming that the system load $\beta$, the input distribution $p_{\boldsymbol{X}}$ and the received SIR distribution $P_\Gamma$ are known, we give the large-system limit of the multiuser efficiency and the spectral efficiency.

Given a scalar $\alpha > 0$, consider a Gaussian channel as depicted in Fig. 2:

$$Z = \sqrt{\alpha}\, X + W \quad (23)$$

where $W$ is a unit Gaussian random variable independent of $X$. We assume that the input $X$ takes the distribution $p_X$. Given that $Z$ is received, we can define an estimate of $X$ as

$$\tilde{X} = \mathsf{E}\{X | Z; \alpha\} \quad (24)$$

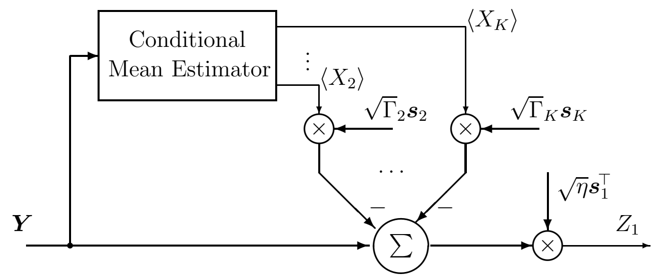


Figure 3: A canonical interference canceler equivalent to the conditional mean estimator.

which is the mean value of $X$ conditioned on $Z$. It is useful here to define the mean square error

$$\mathcal{E}(\alpha) = \mathsf{E}\left\{ \left(\tilde{X} - X\right)^2 \middle| \, \alpha \right\}. \quad (25)$$

In fact, $\tilde{X}$ is the optimal function of $Z$ that minimizes the mean square error. We have the following theorems.

**Theorem 1** *In the large-system limit, the distribution of the conditional mean estimator output $\tilde{X}_k$ of channel (1) conditioned on $X_k = x$ being transmitted is the same as that of the conditional mean estimate $\tilde{X}$ of channel (23) conditioned on $X = x$ being transmitted with $\alpha = \eta\Gamma_k$, where the multiuser efficiency $\eta$ is a solution to a fixed-point equation:*

$$\eta + \eta \cdot \beta \cdot \mathsf{E}\{\Gamma \cdot \mathcal{E}(\eta\Gamma)\} = 1 \quad (26)$$

*where the expectation is taken over the SIR distribution $P_\Gamma$.*

**Theorem 2** *In the large-system limit, the channel capacity for a user with input distribution $p_X$ and SIR $\Gamma$ under the conditional mean estimator and single-user decoding is equal to the input-output mutual information across the single-user Gaussian channel (23) with $\alpha = \eta\Gamma$ where $\eta$ is the multiuser efficiency given by Theorem 1. In case of multiple solutions to (26), $\eta$ is chosen as the smallest one.*

**Theorem 3** *The gain of joint decoding over separate decoding in the large-system spectral efficiency of CDMA is*

$$\frac{1}{2}(\eta - 1 - \log\eta) = \mathsf{D}\left(\mathcal{N}(0,\eta) \,||\, \mathcal{N}(0,1)\right). \quad (27)$$

Remarks: The conditional mean estimator plays an important role in quantifying the spectral efficiency of CDMA. Theorem 1 reveals that each single-user channel resulted from applying the conditional mean estimator to a multiuser channel is equivalent to a degraded Gaussian channel as depicted in Fig. 2. The multiuser efficiency is a number in $[0, 1]$ associated with CDMA as a solution to a fixed-point equation. The effective energy is the input energy times the same multiuser efficiency for all users. The single-user channel capacity is simply the mutual information across the degraded Gaussian channel under the input distribution $p_X$, as concluded in Theorem 2. The total spectral efficiency under joint decoding is closely related to that under separate decoding. The difference, as identified in Theorem 3, is simply a divergence in between two Gaussian distributions.

Our results allow a simple interpretation: The performance averaged over spreading sequences is equivalent to an equilibrium of the multiuser interference game. This can be best illustrated by introducing a canonical interference canceler as shown in Fig. 3. Suppose that the conditional mean estimates (denoted by $\langle X_k \rangle$) are available for all users but user 1. A decision statistic for user 1 is generated by first subtracting the reconstructed interferences using the estimates and then matched filtering with respect to user 1's spreading sequence,

$$\frac{1}{\sqrt{\eta}} Z_1 = \sqrt{\Gamma_1}\, X_1 + \sum_{k=2}^{K} \boldsymbol{s}_1^\top \boldsymbol{s}_k \sqrt{\Gamma_k}\, (X_k - \langle X_k \rangle) + W_1 \qquad (28)$$

where $W_1$ is a unit Gaussian random variable. By Theorem 1, the variance of the multiple access interference, assuming the estimation errors are uncorrelated, is found as a weighted sum of $\mathcal{E}(\eta\Gamma_k)$. The resulting SIR for user 1 is therefore

$$\frac{\Gamma_1}{1 + \beta \cdot \mathsf{E}\left\{\Gamma \cdot \mathcal{E}(\eta\Gamma)\right\}}. \qquad (29)$$

By the fixed-point equation in Theorem 1, (29) is equal to $\eta\Gamma_1$, which is exactly the same as that of a conditional mean estimator. We have thus shown that the conditional mean estimate for one user can be regarded as the output of an interference canceler using the conditional mean estimates of all other users. The multiuser efficiency is such that an equilibrium is achieved, so that every user enjoys the same efficiency; otherwise the users with worse efficiency may benefit from users with better efficiency until an equilibrium is reached.

Using Theorems 1–3, we can easily reproduce previously known capacity results under Gaussian and binary inputs. Before proceeding, we give an explicit expression for the mean square error (25).

The degraded Gaussian channel (23) is characterized by

$$p_{Z|X}(z;\alpha \,|\, x) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(z - \sqrt{\alpha}\, x)^2\right]. \qquad (30)$$

In addition, we define

$$q(z;\alpha) = \mathsf{E}\left\{X \cdot p_{Z|X}(z;\alpha \,|\, X)\right\}, \quad i = 0, 1, \ldots. \qquad (31)$$

Clearly, the decision function in Fig. 2 can be written as

$$\mathsf{E}\left\{X | Z = z; \alpha\right\} = \frac{q(z;\alpha)}{p_Z(z;\alpha)}. \qquad (32)$$

Thus, the mean square error is expressed as

$$\mathcal{E}(\alpha) = 1 - \int \frac{[q(z;\alpha)]^2}{p_Z(z;\alpha)}\, \mathrm{d}z. \qquad (33)$$

### A Gaussian Inputs

The Gaussian prior is known to give the maximum of the mutual information, i.e., the power constrained channel capacity. Let the prior distribution be

$$p_{\boldsymbol{X}}^{(\mathrm{n})}(\boldsymbol{x}) = \prod_{k=1}^{K} \left[\frac{1}{\sqrt{2\pi}} e^{-\frac{x_k^2}{2}}\right], \qquad (34)$$

where we use a superscript $(n)$ to denote normal distribution. The conditional mean estimator is merely a linear amplifier and mean square error is easily shown to be

$$\mathcal{E}^{(\mathrm{n})}(\alpha) = \frac{1}{1+\alpha}. \qquad (35)$$

By Theorem 1, one finds that the multiuser efficiency satisfies the Tse-Hanly equation [5, 2]

$$\eta + \beta \cdot \mathsf{E}\left\{\frac{\eta\Gamma}{1 + \eta\Gamma}\right\} = 1, \qquad (36)$$

which has a unique positive solution $\eta^{(\mathrm{n})}$. By Theorem 2, the single-user channel capacity for a user of SIR $\Gamma$ is

$$\mathsf{C}^{(\mathrm{n})}(\Gamma) = \frac{1}{2} \log\left(1 + \eta^{(\mathrm{n})}\Gamma\right). \qquad (37)$$

By Theorem 3, the total spectral efficiency is expressed in terms of the multiuser efficiency,

$$\mathsf{C}_{\mathrm{joint}}^{(\mathrm{n})} = \frac{\beta}{2} \mathsf{E}\left\{\log\left(1 + \eta^{(\mathrm{n})}\Gamma\right)\right\} + \frac{1}{2}\left(\eta^{(\mathrm{n})} - 1 - \log \eta^{(\mathrm{n})}\right), \qquad (38)$$

as first derived by Shamai and Verdú for fading channels [6].

### B Binary Inputs

It is practically appealing to know the spectral efficiency where the input symbols are constrained to be antipodally modulated as in the popular BPSK and QPSK. Equally probable $\pm 1$'s maximizes the mutual information in this case,

$$p_{\boldsymbol{X}}^{(\mathrm{b})}(\boldsymbol{x}) = 2^{-K}, \;\; \forall \boldsymbol{x} \in \{-1, 1\}^K. \qquad (39)$$

It is not difficult to show that

$$\mathcal{E}^{(\mathrm{b})}(\alpha) = 1 - \int \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \tanh\left(\alpha - z\sqrt{\alpha}\right) \mathrm{d}z. \qquad (40)$$

The multiuser efficiency, $\eta^{(\mathrm{b})}$, is a solution to

$$\eta + \eta\, \beta \cdot \mathsf{E}\left\{\Gamma - \Gamma \int \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \tanh\left(\eta\Gamma - z\sqrt{\eta\Gamma}\right) \mathrm{d}z\right\} = 1, \qquad (41)$$

as was a generalization by Guo and Verdú [8] of an earlier result on equal SIRs due to Tanaka [7]. The single-user channel capacity for a user with SIR $\Gamma$ is the same as that obtained by Müller and Gerstacker [14]

$$\mathsf{C}^{(\mathrm{b})}(\Gamma) = \eta^{(\mathrm{b})}\,\Gamma - \int \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \log\cosh\left(\eta^{(\mathrm{b})}\Gamma - z\sqrt{\eta^{(\mathrm{b})}\Gamma}\right) \mathrm{d}z. \qquad (42)$$

The total spectral efficiency of the CDMA channel with binary inputs is immediate by Theorem 3, also a generalization of Tanaka's inexplicit result in [7] by Guo and Verdú [8].

## V. Numerical Results

We plot the multiuser efficiency and the spectral efficiency as a function of the SIR. Two input distributions are considered, namely, Gaussian inputs and binary inputs. We consider equal SIR for all users (perfect power control) only.

In Fig. 4 we plot the multiuser efficiency as a function of the SIR. A system load of $\beta = 1$ and $\beta = 3$ are considered. We find the multiuser efficiency under Gaussian inputs decrease from 1 to 0 as the SIR increases. This can be easily checked by inspecting the Tse-Hanly equation (36). The multiuser efficiency is not monotonic for binary inputs. Under a system load of $\beta = 1$, the multiuser efficiency converges to 1 for both diminishing SIR and infinite SIR. The case of $\beta = 3$ is more curious. Multiple solutions to the fixed-point equation (41)

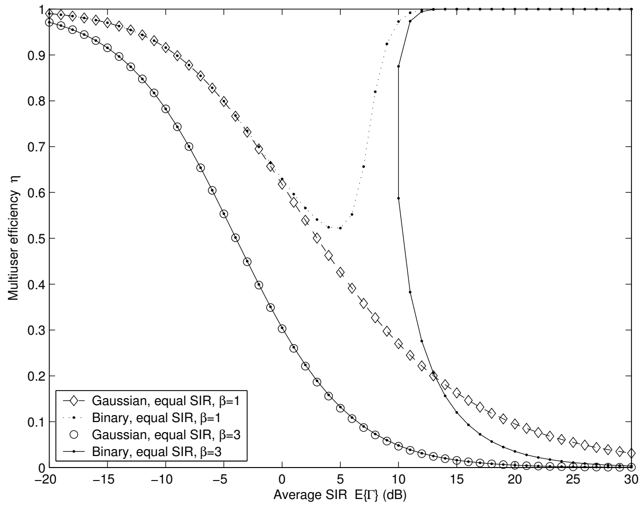


Figure 4: Multiuser efficiency vs. SIR.

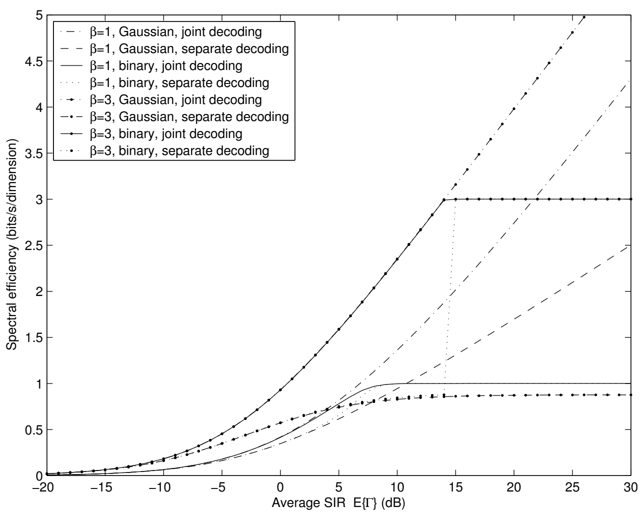


Figure 5: Spectral efficiency vs. SIR.

coexist for an average SIR of 11 dB or higher. This is called phase transition in statistical physics. This can be shown by taking the limit $\Gamma \to \infty$ in (41). Essentially, if $\eta\Gamma \to \infty$, then $\eta \to 1$; while if $\eta\Gamma \to \tau$ where $\tau$ is the solution to

$$\tau \int \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} [1 - \tanh(\tau - z\sqrt{\tau})] \, \mathrm{d}z = \frac{1}{\beta}, \tag{43}$$

then $\eta \to 0$. If $\beta > 2.085$, there exists a solution to (43) so that two modes coexist for large SIR. It is interesting that Gaussian inputs result in about the same multiuser efficiency as binary inputs but without phase transition.

In Fig. 5 we plot the overall spectral efficiency as a function of the input SIR. The spectral efficiency under both joint decoding and separate decoding are plotted. It is clear that the loss in spectral efficiency due to separate decoding is not significant for binary inputs while quite significant for Gaussian inputs in the case of $\beta = 1$. In the case of $\beta = 3$, as a result of phase transition, one observes a jump to saturation in the spectral efficiency under binary inputs. On the other hand, Gaussian inputs suffer great loss if decoded separately, where the spectral efficiency under separate decoding saturates well below that of binary inputs. We can infer that the conditional mean estimator is not efficient in case of dense constellation.

## VI. Conclusion

Assuming an arbitrary input distribution, we give exact expressions for the multiuser efficiency and the spectral efficiency of Gaussian CDMA channels subject to fading in the large-system limit, both under joint and separate decoding. Using the general expressions obtained in this paper, it is straightforward to particularize the results to any input constellation, which can be useful for the design and analysis of CDMA as well as MIMO channels.

## References

[1] A. J. Grant and P. D. Alexander, "Random sequence multisets for synchronous code-division multiple-access channels," *IEEE Trans. Inform. Theory*, vol. 44, pp. 2832–2836, Nov. 1998.

[2] S. Verdú and S. Shamai, "Spectral efficiency of CDMA with random spreading," *IEEE Trans. Inform. Theory*, vol. 45, pp. 622–640, March 1999.

[3] S. Verdú, "Capacity region of Gaussian CDMA channels: The symbol-synchronous case," in *Proceedings 24th Allerton Conference on Communication, Control and Computing*, pp. 1025–1034, Oct. 1986.

[4] J. W. Silverstein and Z. D. Bai, "On the empirical distribution of eigenvalues of a class of large dimensional random matrices," *Journal of Multivariate Analysis*, vol. 54, pp. 175–192, Aug. 1995.

[5] D. N. C. Tse and S. V. Hanly, "Linear multiuser receivers: effective interference, effective bandwidth and user capacity," *IEEE Trans. Inform. Theory*, vol. 45, pp. 622–640, March 1999.

[6] S. Shamai and S. Verdú, "The impact of frequency-flat fading on the spectral efficiency of CDMA," *IEEE Trans. Inform. Theory*, vol. 47, pp. 1302–1327, May 2001.

[7] T. Tanaka, "A statistical mechanics approach to large-system analysis of CDMA multiuser detectors," *IEEE Trans. Inform. Theory*, vol. 48, pp. 2888–2910, Nov. 2002.

[8] D. Guo and S. Verdú, "Multiuser detection and statistical mechanics," in *Communications, Information and Network Security* (V. Bhargava, H. V. Poor, V. Tarokh, and S. Yoon, eds.), ch. 13, pp. 229–277, Kluwer Academic Publishers, 2002.

[9] R. R. Müller and W. H. Gerstacker, "On the capacity loss due to separation of detection and decoding," *IEEE Trans. Inform. Theory*, 2002. submitted.

[10] R. R. Müller, "On channel capacity, uncoded error probability, ML-detection and spin glasses," in *Proceedings Workshop on Concepts in Information Theory*, pp. 79–81, Breisach, Germany, June 2002.

[11] S. F. Edwards and P. W. Anderson, "Theory of spin glasses," *Journal of Physics F: Metal Physics*, vol. 5, pp. 965–974, May 1975.

[12] H. Nishimori, *Statistical Physics of Spin Glasses and Information Processing: An Introduction.* Number 111 in International Series of Monographs on Physics, Oxford University Press, 2001.

[13] D. Guo and S. Verdú, "CDMA spectral efficiency: Asymptotics via statistical physics," *IEEE Trans. Inform. Theory*, 2003. to be submitted.

[14] R. R. Müller and W. Gerstacker, "On the capacity loss due to separation of detection and decoding in large CDMA systems," in *Proceedings IEEE Information Theory Workshop*, p. 222, Bangalore, India, Oct. 2002.