Kolmogorov Complexity – Assignment 2

1. Show that if A and B are prefix-free sets then so is

$$AB = \{xy \mid x \in A \text{ and } y \in B\}.$$

2. Show that

   (a) $\sum_{x\in\Sigma^*} 2^{-C(x)}$ is unbounded.

   (b) $\sum_{x\in\Sigma^*} 2^{-K(x)} \leq 1$.

3. Show the following converse to Kraft's inequality: Given m positive integers $\ell_1 \leq \ell_2 \leq \ldots \leq \ell_m$ such that

$$\sum_{i=1}^{m} \frac{1}{2^{l_i}} \leq 1$$

   show that there exists a (binary) prefix-free code of m strings whose codeword lengths are exactly $l_1, l_2, \ldots, l_m$.

   [Hint: For $i = 1$ to m, let x_i be the first code-word of length l_i such that x_i does not violate the prefix-free condition with the codewords $x_1, \ldots, x_{i-1}$.]

4. Show there is a c such that for all x,

$$C(x) \leq K(x|\ C(x)) + c.$$

5. (Challenge Problem) Show that for every n there is an x with $|x| = n$ such that for all but finitely many y,

$$C(x|y) \leq C(x) - \frac{n}{2}.$$