Kolmogorov Complexity – Assignment 3

1. Let U be a universal programming language. Let α be an infinite 0-1 string.
   (a) Describe a program P that on input α, $P(\alpha)$ outputs $U(p)$ if p is a prefix of α. Why can only one such p cause $U(p)$ to output?
   (b) Show that $\Pr(P(\alpha) \text{ halts}) = \Omega$ for Chatin's Ω. This is why Ω is sometimes called the "Halting Probability."

2. Show that for all n

$$\sum_{x \in \Sigma^n} R(x) \leq cR(1^n)$$

   where $R(x) = 2^{-K(x)}$.

3. Show that for infinitely many x

$$K(x) \geq |x| + K(|x|) - c$$

4. Let $R = \{x \mid C(x) \geq |x|\}$. Show that given access to R as an oracle (having a magic device that can answer questions of the form "Is y in R?") you can solve the halting problem.

   Hint: For a program p consider the amount of time it takes to enumerate all the non-random strings of length $2|p|$.

5. An arithmetic progression is a series of numbers $x_1, \ldots, x_k$ such that for all i, $1 \leq i < k$, $x_{i+1} = x_i + b$ for some constant $b > 0$.

   Let $X(c, k)$ be the largest m such that there is a coloring of the integers $1, \ldots, m$ with c colors such that no arithmetic progression of length k is monochromatic. Use Kolmogorov complexity to show a lower bound for $X(c, k)$.