Kolmogorov Complexity – Assignment 5

Fix strings x and y of length n. Consider S to be the set of strings (u, v) such that

- $K(uv) \leq K(xy)$
- $K(u) \leq K(x)$
- $K(v) \leq K(y)$
- $K(u|v) \leq K(x|y)$, and
- $K(v|u) \leq K(y|x)$

Let $S_1 = \{u \mid (u, v) \in S \text{ for some } v\}$ and $S_2 = \{v \mid (u, v) \in S \text{ for some } u\}$. Note $(x, y) \in S$, $x \in S_1$ and $y \in S_2$.
Consider the random variables A and B that one gets from drawing (u, v) uniformly at random from S and letting A be u and B be v.
We wish to show $H(A) = K(x) \pm O(\log n)$.

1. Show for any random variable X with finite domain D
   (a) $H(X) \leq \log |D|$.
   (b) $H(X) \geq -\log p$ if for every y in D, $p \geq \Pr(X = y)$.
2. Show $|S| \leq 2^{K(xy)+1}$ and $|S_1| \leq 2^{K(x)+1}$.
3. Fix a value $u' \in S_1$. Show that
$$\Pr(A = u') = \frac{|\{(u', v) \mid (u', v) \in S\}|}{|S|}.$$
4. Show that
$$\log(|\{(u', v) \mid (u', v) \in S\}|) \leq K(y|x) = K(xy) - K(x) + O(\log n).$$
5. Show $H(A) = K(x) \pm O(\log n)$