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Lecture note of EECS 395 Winter 2008 (Prof. Fortnow) taken by Bach Ha

Normal reduction

- $A \leq_M^P B$ (P: Polynomial time, M: many 1s)
- $x \in A \Leftrightarrow f(x) \in B$ for some $f \in FP$

Example of Tautology problem

- Tautology is like SAT but for all assignments $TAUT = \{\phi \mid \phi \text{ is true for all assignment}\}$
- ϕ is a tautology $\Leftrightarrow$ $\not\phi$ is not satisfiable
- Hence being able to solve SAT means being able to solve $TAUT$
- However, we cannot say $TAUT \leq_M^P SAT$ since we need to switch sign of the answer

Example of Exact Clique problem

- $EXACTCLIQUE = \{(G, k) \mid \text{The largest clique of } G \text{ of size } k\}$
- This can be solve by seeing whether or not there is a clique of size k and no clique of size $k+1$.
- But again, we cannot use this kind of reduction because we can only ask one question, here two questions were asked.

Oracle Turing Machine

- Sort of a standard TM that has input, work tapes, and special oracle tape and 3 special state, $q_?$, q_Y and q_N.
- M writes down a query on the oracle tape, it enters $q_?$ and "magically" in one step goes to either q_Y or q_N.
- Let $A \in \Sigma^*$, and $M^A(x)$ with oracle A and input x. When M writes x on the oracle tape, it enters $q_?$, if $x \in A$ then it goes to q_Y or else go to q_N

- $M^{EXACTCLIQUE}(G,k)$ accepts if $(G,k) \in CLIQUE$ and $(G,k+1) \notin CLIQUE$. THis machine accepts $\Leftrightarrow$ it is an exact clique.

Broader notion of reduction: Turing reduction

- $A \leq_T^P B$ (reads A is poly time Turing reducible to B) iff there is a poly time oracle TM M such that $x \in A \Leftrightarrow M^B(x)$ accepts.
- Example: We say $EXACTCLIQUE \leq_T^P CLIQUE$, the reverse is true by consider all cases
- Lemma: If $A \leq_T^P B$ and $B \in P$ then $A \in P$
- This is probably not true for NP.

Some more complexity classes (quantum hiearchy)

- We want to put some more complexity classes in between $NP/co-PN$ and $PSPACE$ hierarchy graph.
- Define P^A is a set of languages such that there exists a poly-time oracle TM M such that $x \in L \Leftrightarrow M^A(x)$ accepts. Similar with NP^A.
- Define $\Delta_0^P = \Sigma_0^P = \Pi_0^P = P$
- $P^C = \bigcup_{A \in C} P^A$. Hence $EXACTCLIQUE \in P^{NP}$ since $CLIQUE \in NP$ and $EXACTCLIQUE \in P^{CLIQUE}$
- Let $\Delta_{k+1}^P = P^{\Sigma_k^P}$, $\Sigma_{k+1}^P = NP^{\Sigma_k^P}$, $\Pi_{k+1}^P = co - NP^{\Sigma_k^P} = co - \Sigma_{k+1}^P$, $PH = \bigcup_n \Sigma_k^P$
- Some notes:
  - $\Delta_1^P = P$,
  - $\Sigma_1^P = NP^P = NP$,
  - $PH \in PSAPCE$ since: (Using induction)
    * $\Sigma_0^P = P \in PSAPCE$
    * $\Sigma_{k+1}^P = NP^{\Sigma_k^p} \leq NP^{PSPACE} \leq NSPACE = PSPACE$
- $\Sigma_k^P \in \Sigma_{k+1}^P$. Similarly, $\Pi_k^p \leq \Sigma_{k+1}^P$:
  - If $L \in \Pi_k^P$, that means $\overline{L} \in \Sigma_k^P$
  - $\Sigma_{k+1}^P = NP^{\Sigma_k^P}$

- $M^{\overline{L}}(x)$ accepts if $x \notin \overline{L}$.

- $\Sigma_k^P \leq P^{\Sigma_k^P}$,
- $\Delta_{k+1}^P \leq \Sigma_{k+1}^P \leq \Pi_{k+1}^P$.
- This is called quantum hierarchy

- Is it infinite?
- The following are equivalent:
  - $\Sigma_k^P = PH$ ($\Sigma_k^P = \Sigma_l^P$ for all $l \geq k$),
  - $\Sigma_k^P = \Pi_k^P$,
  - $\Sigma_k^P = \Delta_{k+1}^P$,
  - $\Sigma_k^P = \Sigma_{k+1}^P$.
- Call above as a, b, c, d, we can show $a \Rightarrow d \Rightarrow\ c \Rightarrow b \Rightarrow a$.