# Differentiated Internet Pricing Using a Hierarchical Network Game Model

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***Abstract*— This paper builds on the model and results of [1], and extends them to the case of differentiated prices, again for the single link case. It introduces a hierarchical network game with one service provider and multiple users of different types, where the service provider is allowed to charge different prices to users of different types. The service provider plays with the users a Stackelberg (leader-follower) game, while among users themselves, they play a Nash game. The paper establishes for a general network with multiple links the existence of a unique Nash equilibrium along with the existence of a unique Stackelberg solution. The economics of providing large capacity and price differentiation is examined especially in the single link case and for a many-user regime. One important result is that optimum price differentiation leads to a more egalitarian distribution of resources at fairer prices and improves the service provider's revenue and network performance. Moreover, the service provider has an incentive to increase the capacity proportionally with the number of additional users admitted.**

***Index Terms*— Internet pricing, Price differentiation, Many-User limit, Stackelberg game, Noncooperative game, Quality-of-Service, Flow control**

## I. Introduction

Control of the Internet has become an important research topic recently, driven by the need to avoid congestion due to the increasing use of network resources and to provide Quality of Service (QoS) guarantees. One branch of network control is routing, which assumes that the total flow of each user is fixed [6], [7]. Another one is flow control under fixed routes. This paper belongs to the latter category and can be viewed as a continuation of the research reported earlier in [1] and [2].

In general, flow control needs to be combined with appropriate pricing. Also, among the users and Internet Service Providers (ISPs), competition and optimization are commonplace. All these problems can be captured well within the framework of game theory [3]. They may be studied from noncooperative [1], [2], [8] or cooperative [9] perspectives. While it has been shown in [9] that all players (users and ISPs) are better off by cooperation, arbitration is needed for this scheme, and hence a large portion of research is focused on the noncooperative framework, including the work here.

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It has been pointed out in [1] that as the number of users and the capacity of the network become large, the probability of queue build-up is small and the economics of providing large capacity from the ISP's point of view becomes a relevant issue, as examined there for a single link network. A more complex linear network model was then studied in [2]. While adopting the models used in [1], [2], this paper extends the network game from the uniform price scenario to a differentiated one and presents some new results. The impact of price differentiation on users, ISPs and performance of the network is studied to meet the requirement for today's Internet to provide differentiated services and QoS guarantees [10], [11].

The paper is organized as follows. In Section II, the problem is formulated for a general network which is modeled as a two-level hierarchical game and the uniqueness of Nash equilibrium is established for fixed prices set by ISP. The study first focuses on a single link network for a special case in Section III and then on a more general single link case in Section IV. In both cases, a comparative study of uniform price and differentiated price cases are conducted, especially in a many-user regime. The paper ends with some discussions and conclusions.

## II. Problem Formulation

### A. General Network

We assume that there is a single service provider, which is addressed as "the network". [1] In a general network model, denote the set of users as $\mathcal{I} = \{1, \cdots, I\}$, and the set of links as $\mathcal{L} = \{1, \cdots, L\}$. Link $l$, $l \in \mathcal{L}$, has a capacity $c_l$. For User $i$, $i \in \mathcal{I}$, $x_i$ is his transmission rate and $\mathcal{L}_i \subseteq \mathcal{L}$ is the set of links $x_i$ traverses. By accessing the network at a rate $x_i$, User $i$ achieves a utility of $w_i \log(1 + k_i x_i)$, $w_i, k_i > 0$. [2] However, he has to pay to the service provider for usage of the network, and we let $p_{li}$ denote the price per unit bandwidth charged to User $i$ for using link $l$, $l \in \mathcal{L}_i$. He also faces a cost $\frac{1}{c_l - \bar{x}_l}$ due to congestion, where $\bar{x}_l =$

[1] The model can be extended to a network with multiple competitive service providers, which will be discussed in the conclusion section of this paper.

[2] In [1], a utility function $w_i \log(1 + x_i)$ is adopted for User $i$, which originates from the concept of proportionally fair resource allocation [4], [5]. Here $k_i$ is introduced for the scalability of $x_i$.

$\sum_{i:l\in\mathcal{L}_i} x_i$. User $i$ has a (net) utility [3]

$$F_i = w_i \log(1+k_i x_i) - \sum_{l\in\mathcal{L}_i} \frac{1}{c_l - \bar{x}_l} - v_i x_i \sum_{l\in\mathcal{L}_i} p_{li} \quad (1)$$

and the network collects the revenue

$$R = \sum_{l\in\mathcal{L}} \sum_{i:l\in\mathcal{L}_i} p_{li} x_i = \sum_{i\in\mathcal{I}} x_i \sum_{l\in\mathcal{L}_i} p_{li}. \quad (2)$$

### B. Two-Level Hierarchical Game

The problem constitutes a two-level hierarchical network game [3]. The upper level is a Stackelberg game with the network being the leader and the users being the followers. In this game, the network determines prices such that the users will respond with certain transmission rates to maximize the total revenue. Usually, the prices can be decided in two ways: a uniform price for any user along any link (fixed-rate pricing) or differentiated prices for different users along different links. The former one is simple and largely adopted currently while the latter has its own advantages, which is one focus of this paper. For convenience, we refer to these two schemes as UniPri and DiffPri, respectively.

At the lower level is an $I$-player noncooperative game: given prices, User $i$ chooses the value of $x_i$ to maximize his utility $F_i$ subject to $x_i \geq 0$ and the capacity constraints $\bar{x}_l < c_l$, $l \in \mathcal{L}$, $i \in \mathcal{I}$. The solution to this is captured in the following theorem:

*Theorem 1:* (Existence of Unique Nash Equilibrium) For each fixed set of prices, the $I$-player noncooperative game admits a unique Nash equilibrium solution.

*Proof:* Following the approach of [1], we add to $F_i$ the quantity

$$\sum_{j\neq i} w_j \log(1+k_j x_j) - \sum_{l\notin\mathcal{L}_i} \frac{1}{c_l - \bar{x}_l} - \sum_{j\neq i} v_j x_j \sum_{l\in\mathcal{L}_j} p_{lj}$$

and obtain a common function for all users:

$$F = \sum_{i\in\mathcal{I}} w_i \log(1+k_i x_i) - \sum_{i\in\mathcal{I}} v_i x_i \sum_{l\in\mathcal{L}_i} p_{li} - \sum_{l\in\mathcal{L}} \frac{1}{c_l - \bar{x}_l}.$$

Since the added quantity for $F_i$ is not related to $x_i$, this will not change the Nash equilibrium. Thus, the original $I$-player game is equivalent to the noncooperative game where all users have a common objective function $F$. Furthermore, we have

$$F_{x_i x_i} = -\frac{w_i k_i^2}{(1+k_i x_i)^2} - \sum_{l\in\mathcal{L}_i} \frac{2}{(c_l - \bar{x}_l)^3} < 0, \quad i \in \mathcal{I},$$

$$F_{x_i x_j} = -\sum_{l\in\mathcal{L}_i\cap\mathcal{L}_j} \frac{2}{(c_l - \bar{x}_l)^3} < 0, \quad i,j \in \mathcal{I},\ j \neq i.$$

Therefore, the *Hessian* matrix of $F$ is negative definite and thus $F$ is strictly concave, which implies that the $I$-player game admits a unique Nash equilibrium. ■

[3]Here we could have included a multiplicative weight parameter in the second term, but instead, without any loss of generality, we include such a parameter, $v_i$, in the third term. This also eases the analysis.

## III. Complete Solution for a Special Single Link Case

### A. Uniform Price

We consider a single link network with a capacity $nc$ shared by $n$ users. A special case with $c = 1$ is studied in this section and the analysis will be extended to a general $c$ in the following one. Denote the set of users as $N := \{1, \cdots, n\}$ and the network charges a uniform price $p$ per unit bandwidth for each user. A complete solution for this is obtained in [1], which we reproduce here since the underlying expressions will be needed in our comparative study later. Given the price $p$, User $i$ determines $x_i$ to maximize his utility

$$F_i = w_i \log(1+x_i) - \frac{1}{n-\bar{x}} - p x_i, \quad i \in N,$$

where $\bar{x} := \sum_{j=1}^n x_j$. A positive Nash equilibrium solution exists if and only if $(F_i)_{x_i} = 0$, $i \in N$, admits a positive solution. If it is so, we have

$$\frac{1}{(nc-\bar{x})^2} + p = \frac{w_i}{1+x_i} = \frac{\bar{w}}{n+\bar{x}}, \quad i \in N, \quad (3)$$

where $\bar{w} := \sum_{j=1}^n w_j$. Now proceeding to the leader's problem where the network chooses the price to maximize its revenue $R = p\bar{x}$, from the one-to-one correspondence between $\bar{x}$ and $p$ in (3), it follows that this is equivalent to maximizing $R$ with respect to $\bar{x}$. It can be easily verified that $R_{\bar{x}\bar{x}} < 0$ and thus $R$ is strictly concave with respect to $\bar{x}$ at $(0, n)$. Furthermore, $R$ is negative unbounded at the upper end of the interval. Then letting $R_{\bar{x}} = 0$, the optimal positive solution involving the throughput, congestion cost, flows, price and revenue per unit bandwidth for the uniform price case is obtained as follows:

$$x^*_{av-u} = 1 - \frac{2}{1+(n^2 w_{av})^{\frac{1}{3}}}, \quad (4)$$

$$d^*_u = \frac{1}{n - n x^*_{av-u}} = \frac{1+(n^2 w_{av})^{\frac{1}{3}}}{2n}, \quad (5)$$

$$x^*_{i-u} = \frac{w_i}{w_{av}}(x^*_{av-u}+1) - 1, \quad i \in N, \quad (6)$$

$$p^*_u = \frac{w_{av}}{2}(1+(n^2 w_{av})^{-\frac{1}{3}}) - \frac{1}{4n^2}(1+(n^2 w_{av})^{\frac{1}{3}})^2, \quad (7)$$

$$r^*_u = p^*_u x^*_{av-u} = \frac{w_{av}}{2} - \frac{3}{4n^2}(n^2 w_{av})^{\frac{2}{3}} + \frac{1}{4n^2}, \quad (8)$$

if and only if

$$w_i > \frac{2(n^2 w_{av})^{\frac{2}{3}} + 2n^2 w_{av}}{4n^2}, \quad \forall\ i \in N, \quad (9)$$

where $w_{av} := \frac{\bar{w}}{n}$ and $x_{av} := \frac{\bar{x}}{n}$.

Asymptotically, with $n$ growing large, the necessary and sufficient condition (9) for the solution to be positive becomes

$$w_i > \frac{w_{av}}{2}, \quad \forall\ i \in N. \quad (10)$$

Under this condition, we have, asymptotically,

$$x^*_{av-u}(n) \sim 1 - 2{w_{av}}^{-\frac{1}{3}} n^{-\frac{2}{3}}, \tag{11}$$

$$d^*_u(n) \sim \frac{1}{2}{w_{av}}^{\frac{1}{3}} n^{-\frac{1}{3}}, \tag{12}$$

$$x^*_{i-u}(n) \sim \frac{2w_i}{w_{av}}(1 - {w_{av}}^{-\frac{1}{3}} n^{-\frac{2}{3}}) - 1, \quad i \in N, \tag{13}$$

$$p^*_u(n) \sim \frac{w_{av}}{2} + \frac{1}{4}{w_{av}}^{\frac{2}{3}} n^{-\frac{2}{3}}, \tag{14}$$

$$r^*_u(n) \sim \frac{w_{av}}{2} - \frac{3}{4}{w_{av}}^{\frac{2}{3}} n^{-\frac{2}{3}}. \tag{15}$$

Generally, condition (9) may not be satisfied. Then, the solution can be obtained in the following way. Order the users such that $w_i > w_j$ only if $i < j$. For any integer $\tilde{n} \le n$, include only the first $\tilde{n}$ users in the game, and carry out the analysis as in the previous case but by replacing $n$ by $\tilde{n}$. Let $n^*$ be the largest such $\tilde{n}$, so that the necessary and sufficient condition for a positive solution is satisfied. It is shown in [1] that the problem admits a unique solution which is the positive solution for the game with the first $n^*$ players appended by $x^*_{i-u} = 0$, $i > n^*$. In other words, only those users with large enough $w_i$'s are admitted with positive transmission rates.

### B. Differentiated Prices

Now the problem remains the same, except that the network may charge different prices for different users. Let $p_i$ be the price per unit bandwidth for User $i$, $i \in N$. The users' utilities are:

$$F_i = w_i \log(1 + x_i) - \frac{1}{n - \bar{x}} - p_i x_i, \quad i \in N.$$

From $(F_i)_{x_i} = 0$, $i \in N$,

$$p_i = \frac{w_i}{1 + x_i} - \frac{1}{(n - \bar{x})^2}, \quad i \in N.$$

Similarly, the revenue $R = \sum_{j=1}^{n} p_j x_j$ is maximized with respect to $(x_1, \cdots, x_n)$. Since the *Hessian* matrix of $R$ is negative definite, $R$ is strictly concave. Furthermore, since $R \downarrow -\infty$ as $\bar{x} \uparrow n$, the above optimization problem admits a unique solution. $R_{x_i} = 0$, $i \in N$, implies

$$\sqrt{\frac{(n - \bar{x})^3}{n + \bar{x}}} = \frac{1 + x_i}{\sqrt{w_i}} = \frac{n + \bar{x}}{\bar{v}^{\frac{1}{2}}}, \quad i \in N,$$

where $\bar{v}^{\frac{1}{2}} := \sum_{j=1}^{n} \sqrt{w_j}$, and then it follows that the optimal solution is:

$$x^*_{av-d} = 1 - \frac{2}{1 + (n v_{av}^{\frac{1}{2}})^{\frac{2}{3}}}, \tag{16}$$

$$d^*_d = \frac{1 + (n v_{av}^{\frac{1}{2}})^{\frac{2}{3}}}{2n}, \tag{17}$$

$$x^*_{i-d} = \frac{\sqrt{w_i}}{v_{av}^{\frac{1}{2}}}(x^*_{av-d} + 1) - 1, \quad i \in N, \tag{18}$$

$$p^*_{i-d} = \sqrt{w_i}\frac{v_{av}^{\frac{1}{2}}}{2}(1 + (n v_{av}^{\frac{1}{2}})^{-\frac{2}{3}}) - \frac{1}{4n^2}(1 + (n v_{av}^{\frac{1}{2}})^{\frac{2}{3}})^2,$$
$$i \in N, \tag{19}$$

$$r^*_d = w_{av} - \frac{1}{2n^2}(n v_{av}^{\frac{1}{2}})^2 - \frac{3}{4n^2}(n v_{av}^{\frac{1}{2}})^{\frac{4}{3}} + \frac{1}{4n^2}, \tag{20}$$

where $v_{av}^{\frac{1}{2}} := \frac{\bar{v}^{\frac{1}{2}}}{n}$. The necessary and sufficient condition for the solution to be positive is

$$w_i > \frac{2(n v_{av}^{\frac{1}{2}})^{\frac{4}{3}} + (n v_{av}^{\frac{1}{2}})^2 + (n v_{av}^{\frac{1}{2}})^{\frac{2}{3}}}{4n^2}, \quad \forall\, i \in N. \tag{21}$$

For large $n$, (21) becomes

$$\sqrt{w_i} > \frac{v_{av}^{\frac{1}{2}}}{2}, \quad \forall\, i \in N. \tag{22}$$

Then it follows that, asymptotically,

$$x^*_{av-d}(n) \sim 1 - 2(v_{av}^{\frac{1}{2}})^{-\frac{2}{3}} n^{-\frac{2}{3}}, \tag{23}$$

$$d^*_d(n) \sim \frac{1}{2}(v_{av}^{\frac{1}{2}})^{\frac{2}{3}} n^{-\frac{1}{3}}, \tag{24}$$

$$x^*_{i-d}(n) \sim \frac{2\sqrt{w_i}}{v_{av}^{\frac{1}{2}}}(1 - (v_{av}^{\frac{1}{2}})^{-\frac{2}{3}} n^{-\frac{2}{3}}) - 1, \quad i \in N, \tag{25}$$

$$p^*_{i-d}(n) \sim \frac{\sqrt{w_i}}{2}(v_{av}^{\frac{1}{2}} + (v_{av}^{\frac{1}{2}})^{\frac{1}{3}} n^{-\frac{2}{3}}) - \frac{1}{4}(v_{av}^{\frac{1}{2}})^{\frac{4}{3}} n^{-\frac{2}{3}},$$
$$i \in N, \tag{26}$$

$$r^*_d(n) \sim w_{av} - \frac{1}{2}(v_{av}^{\frac{1}{2}})^2 - \frac{3}{4}(v_{av}^{\frac{1}{2}})^{\frac{4}{3}} n^{-\frac{2}{3}}. \tag{27}$$

If (21) is not satisfied, then the solution can be obtained in the same way as previously stated for UniPri, i.e. an admission policy can be devised.

### C. Comparison of the Two Pricing Schemes

*1) Comparison of Conditions for the Positive Solution:*

*Theorem 2:* For the special single link case, the users admitted with positive transmission rates for UniPri must also be admitted for DiffPri. In other words, more users with smaller $w_i$'s may be possibly admitted due to price differentiation.

*Proof:* We have the inequality

$$n^2 w_{av} = \sum_{i,j} \frac{w_i + w_j}{2} \ge \sum_{i,j} \sqrt{w_i}\sqrt{w_j} = (n v_{av}^{\frac{1}{2}})^2. \tag{28}$$

Thus, if we can show that $n^2 w_{av} \ge (n v_{av}^{\frac{1}{2}})^{\frac{2}{3}}$, then (9) leads to (21). In fact, if (9) holds, it follows that $n^2 w_{av} > 1$. Hence we consider two possibilities: if $n v_{av}^{\frac{1}{2}} \le 1$, obviously $n^2 w_{av} > (n v_{av}^{\frac{1}{2}})^{\frac{2}{3}}$; if $n v_{av}^{\frac{1}{2}} > 1$, then $(n v_{av}^{\frac{1}{2}})^2 > (n v_{av}^{\frac{1}{2}})^{\frac{2}{3}}$ and we arrive at the same result. Therefore, (21) also holds, and consequently $n v_{av}^{\frac{1}{2}} > 1$. ■

*2) Same Number of Users Admitted:* Based on (28), we can compare the optimal solutions for the two schemes given by (4) to (8) and (16) to (20).

*Throughput:* $x^*_{av-u} \ge x^*_{av-d}$.

*Congestion Cost:*

$$d^*_u \ge d^*_d. \tag{29}$$

*Individual Flows:* $x^*_{i-u}$ given by (6) is "proportional" to $w_i$ and equals $x^*_{av-u}$ if and only if $w_i = w_{av}$. $x^*_{i-d}$ given by (18) depends on $\sqrt{w_i}$ and equals $x^*_{av-d}$ if and only if $\sqrt{w_i} = v^{\frac{1}{2}}_{av}$. Furthermore, we have

$$\begin{cases} x^*_{i-u} > x^*_{i-d} & \text{if } w_i > w_x, \\ x^*_{i-u} = x^*_{i-d} & \text{if } w_i = w_x, \\ x^*_{i-u} < x^*_{i-d} & \text{if } w_i < w_x, \end{cases} \quad (30)$$

where

$$w_x := \frac{{w_{av}}^2 (x^*_{av-d}+1)^2}{(v^{\frac{1}{2}}_{av})^2 (x^*_{av-u}+1)^2} = \frac{{w_{av}}^{\frac{4}{3}} [1+(n^2 w_{av})^{\frac{1}{3}}]^2}{(v^{\frac{1}{2}}_{av})^{\frac{2}{3}} [1+(n v^{\frac{1}{2}}_{av})^{\frac{2}{3}}]^2}. \quad (31)$$

*Prices:*

$$\begin{cases} p^*_u < p^*_{i-d} & \text{if } w_i > w_p, \\ p^*_u = p^*_{i-d} & \text{if } w_i = w_p, \\ p^*_u > p^*_{i-d} & \text{if } w_i < w_p, \end{cases} \quad (32)$$

where

$$w_p := \frac{1}{\{v^{\frac{1}{2}}_{av}[1+(n v^{\frac{1}{2}}_{av})^{-\frac{2}{3}}]\}^2} \{\, w_{av}[1+(n^2 w_{av})^{-\frac{1}{3}}] - \frac{1}{2n^2}[1+(n^2 w_{av})^{\frac{1}{3}}]^2 + \frac{1}{2n^2}[1+(n v^{\frac{1}{2}}_{av})^{\frac{2}{3}}]^2 \,\}^2. \quad (33)$$

*Remarks on $w_x$ and $w_p$:* First, as a direct result of (28), $w_x \geq w_{av}$. Then note that

$$\sqrt{w}_{\max} v^{\frac{1}{2}}_{av} \geq w_{av} \geq \sqrt{w}_{\min} v^{\frac{1}{2}}_{av}. \quad (34)$$

Therefore, $w_x \leq [{w_{av}}^{\frac{4}{3}} / (v^{\frac{1}{2}}_{av})^{\frac{2}{3}}] \cdot [(n^2 w_{av})^{\frac{2}{3}} / (n v^{\frac{1}{2}}_{av})^{\frac{4}{3}}] = (w_{av}/v^{\frac{1}{2}}_{av})^2 \leq w_{\max}$. On the other hand, comparing $w_p$ with $w$ is equivalent to comparing $(n^2 w_{av})[1+(n^2 w_{av})^{-\frac{1}{3}}] + \frac{1}{2}[1+(n v^{\frac{1}{2}}_{av})^{\frac{2}{3}}]^2$ with $(n\sqrt{w})(n v^{\frac{1}{2}}_{av})[1+(n v^{\frac{1}{2}}_{av})^{-\frac{2}{3}}] + \frac{1}{2}[1+(n^2 w_{av})^{\frac{1}{3}}]^2$. By substituting $w_x$ and $(v^{\frac{1}{2}}_{av})^2$ for $w$ in the above formula, respectively, and using (28) and (34), we can easily verify that $w_x \geq w_p \geq (v^{\frac{1}{2}}_{av})^2$. In summary,

$$w_{\max} \geq w_x \geq w_p, w_{av} \geq (v^{\frac{1}{2}}_{av})^2 \geq w_{\min}. \quad (35)$$

*Revenue:* $r^*_u \leq r^*_d$.

*Individual Utilities:* Intuitively, for those users with small $w_i$'s, by price differentiation, both the congestion cost and the individual prices get decreased, which is beneficial to such users and thus increases their individual utilities as a result. For all the other users, although the congestion decreases likewise, their individual prices get higher and the larger $w_i$ is, the more the difference between $p^*_{i-d}$ and $p^*_u$ becomes. There is a tradeoff between the decrease of congestion and the increases of individual prices.

Detailed analysis is as follows. For some $i$ such that $w_p \geq w_i \geq w_{\min}$, given the optimal solution for DiffPri, let User $i$'s flow $x_{i-d}$ be $x^*_{i-u}$ instead of $x^*_{i-d}$ and all the other users' flows remain unchanged; then his utility becomes:

$$F_{i-d}(x_{i-d} = x^*_{i-u}) = w_i \log(1+x^*_{i-u}) - \frac{1}{n - (n x^*_{av-d} - x^*_{i-d} + x^*_{i-u})} - p^*_{i-d} x^*_{i-u}.$$

Since $x^*_{i-u} \leq x^*_{i-d}$ and $x^*_{av-u} \geq x^*_{av-d}$, consequently $n x^*_{av-u} \geq n x^*_{av-d} - x^*_{i-d} + x^*_{i-u}$. Also, $p^*_u \geq p^*_{i-d}$. Hence, $F^*_{i-u} \leq F_{i-d}(x_{i-d} = x^*_{i-u})$. On the other hand, by the definition of Nash equilibrium, $F_{i-d}(x_{i-d} = x^*_{i-u}) \leq F^*_{i-d}$. Therefore, $F^*_{i-u} \leq F^*_{i-d}$.

As a conclusion, there exists a threshold $w_F$ (though the explicit expression is hard to be obtained) such that

$$w_{\max} \geq w_F \geq w_p, \quad (36)$$

and

$$\begin{cases} F^*_{i-u} > F^*_{i-d} & \text{if } w_i > w_F, \\ F^*_{i-u} = F^*_{i-d} & \text{if } w_i = w_F, \\ F^*_{i-u} < F^*_{i-d} & \text{if } w_i < w_F. \end{cases} \quad (37)$$

*3) More Users Admitted with Price Differentiation:* Generally, more users can be admitted for DiffPri than for UniPri. Order all the users such that $w_i > w_j$ only if $i < j$. Let $n$ be the largest integer such that the first $n$ users are admitted for UniPri, and $\hat{n}$ be the largest integer for DiffPri. Assume $n < \hat{n}$. Also note that when only the first $n$ users are present, by applying DiffPri, all of them will still be admitted. We denote the above three cases as UniPri($n$), DiffPri($\hat{n}$) and DiffPri($n$), respectively. DiffPri($n$) is introduced to help compare the other two cases from the previous results obtained for UniPri($n$) and DiffPri($n$). The notation for DiffPri($\hat{n}$) is the same as for DiffPri($n$), except with a hat above.

*Throughput:* Obviously,

$$1 < n v^{\frac{1}{2}}_{av} < \hat{n} \hat{v}^{\frac{1}{2}}_{av} \text{ and } v^{\frac{1}{2}}_{av} \geq \hat{v}^{\frac{1}{2}}_{av}. \quad (38)$$

Then by (16),

$$x^*_{av-d} < \hat{x}^*_{av-d}, \quad (39)$$

though we do not know how $x^*_{av-u}$ compares with $\hat{x}^*_{av-d}$.

*Congestion Cost:* By (38) and the fact that $n < \hat{n}$, we have $2n[1+(\hat{n}\hat{v}^{\frac{1}{2}}_{av})^{\frac{2}{3}}] < 2\hat{n}[1+(n v^{\frac{1}{2}}_{av})^{\frac{2}{3}}]$. Then from (17) and (29), $d^*_u \geq d^*_d > \hat{d}^*_d$, which means that users will always be better off in congestion cost with differentiated prices.

*Individual Flows:* From (18), (38) and (39), $x^*_{i-d} < \hat{x}^*_{i-d}$, $1 \leq i \leq n$. Thus, the same pattern as (30) can be expected for $x^*_{i-u}$ versus $\hat{x}^*_{i-d}$, except that the threshold will get larger.

*Revenue:* Note that the link capacity is $n$ for UniPri($n$) and DiffPri($n$) and $\hat{n}$ for DiffPri($\hat{n}$). Thus, instead of comparing the revenue per unit capacity, here we compare the total revenue which the network tries to maximize.

It is hard, if not impossible, to compare $R^*_d = n \cdot r^*_d$ with $\hat{R}^*_d = \hat{n} \cdot \hat{r}^*_d$ from (20). However, if we let $R^{'*}_d$ be the optimal revenue for the network with only a capacity of $n$ but all the $\hat{n}$ users presented, then $\hat{R}^*_d \geq R^{'*}_d$ because of a larger capacity and $R^{'*}_d \geq R^*_d$ because of more users. Thus, $R^*_u \leq R^*_d \leq \hat{R}^*_d$. The network will be better off under price differentiation.

*4) Asymptotic Analysis:* For UniPri($n$) and DiffPri($\hat{n}$), the analysis of asymptotic behavior will lead to a better understanding. As $n \to \infty$ and $\hat{n} \to \infty$, we denote the limits of (10) to (15) and (22) to (27) by adding a '$\sim$' above, correspondingly.

*Throughput:* As $n \uparrow \infty$, $\tilde{x}^*_{av-u} \uparrow 1$; as $\hat{n} \uparrow \infty$, $\tilde{x}^*_{av-d} \uparrow 1$. Thus, $\tilde{x}^*_{av-u} \cong \tilde{x}^*_{av-d}$. Furthermore, $\tilde{\bar{x}}^*_u \sim n$ as $n \uparrow \infty$ and $\tilde{\bar{x}}^*_d \sim \hat{n}$ as $\hat{n} \uparrow \infty$. So $\tilde{\bar{x}}^*_u \leq \tilde{\bar{x}}^*_d$.

*Congestion Cost:* It follows from (28) and (38) that

$$w_{av} \geq (\hat{v}_{av}^{\frac{1}{2}})^2. \tag{40}$$

Thus, $\tilde{d}^*_u \geq \tilde{\tilde{d}}^*_d$.

*Individual Flows:* $\tilde{x}^*_{i-u} \uparrow \frac{2w_i}{w_{av}} - 1$ as $n \uparrow \infty$ and $\tilde{x}^*_{i-d} \uparrow \frac{2\sqrt{w_i}}{\hat{v}_{av}^{\frac{1}{2}}} - 1$ as $\hat{n} \uparrow \infty$. Therefore,

$$\begin{cases} \tilde{x}^*_{i-u} > \tilde{x}^*_{i-d} & \text{if } w_i > \tilde{w}, \\ \tilde{x}^*_{i-u} = \tilde{x}^*_{i-d} & \text{if } w_i = \tilde{w}, \\ \tilde{x}^*_{i-u} < \tilde{x}^*_{i-d} & \text{if } w_i < \tilde{w}, \end{cases} \tag{41}$$

where

$$\tilde{w} := \left(w_{av}/\hat{v}_{av}^{\frac{1}{2}}\right)^2 \geq w_{av}. \tag{42}$$

*Prices:* $\tilde{p}^*_u \downarrow \frac{w_{av}}{2}$ as $n \uparrow \infty$ and $\tilde{p}^*_{i-d} \downarrow \frac{\sqrt{w_i}}{2}\hat{v}_{av}^{\frac{1}{2}}$ as $\hat{n} \uparrow \infty$. Again,

$$\begin{cases} \tilde{p}^*_u < \tilde{p}^*_{i-d} & \text{if } w_i > \tilde{w}, \\ \tilde{p}^*_u = \tilde{p}^*_{i-d} & \text{if } w_i = \tilde{w}, \\ \tilde{p}^*_u > \tilde{p}^*_{i-d} & \text{if } w_i < \tilde{w}. \end{cases} \tag{43}$$

Although $w_x$ in (31) is greater than $w_p$ in (33) in general, the two thresholds share the same limit $\tilde{w}$ as the number of users goes to infinity.

*Revenue:* $\tilde{r}^*_u \uparrow \frac{w_{av}}{2}$ as $n \uparrow \infty$ and $\tilde{r}^*_d \uparrow \hat{w}_{av} - \frac{1}{2}(\hat{v}_{av}^{\frac{1}{2}})^2$ as $\hat{n} \uparrow \infty$. If $n \cong \hat{n}$, then $w_{av} \cong \hat{w}_{av}$. By (40), $\tilde{r}^*_u \leq \tilde{r}^*_d$.

*Individual Utilities:* From the above results, we know immediately that if $w_i < \tilde{w}$, $\tilde{F}^*_{i-u} < \tilde{F}^*_{i-d}$ and if $w_i = \tilde{w}$, $\tilde{F}^*_{i-u} \leq \tilde{F}^*_{i-d}$. Therefore,

$$\begin{cases} \tilde{F}^*_{i-u} > \tilde{F}^*_{i-d} & \text{if } w_i > \tilde{w}_F, \\ \tilde{F}^*_{i-u} = \tilde{F}^*_{i-d} & \text{if } w_i = \tilde{w}_F, \\ \tilde{F}^*_{i-u} < \tilde{F}^*_{i-d} & \text{if } w_i < \tilde{w}_F, \end{cases} \tag{44}$$

where $\tilde{w}_F \geq \tilde{w}$.

### D. Recap

As a conclusion, we can deduce the following general features for the network problem from the foregoing analysis, and particularly from the asymptotic analysis. First, the total flow is higher under DiffPri than UniPri, and thus congestion is alleviated. Also, price differentiation is beneficial to the network for the improved revenue as well as to those users with small $w_i$'s for the reduced prices and increased flows. However, for those users with large $w_i$'s, their utilities get decreased by price differentiation.

Second, both for UniPri and DiffPri, as the number of admitted users increases, the throughput and the flows increase, the congestion is reduced, the prices decrease and the revenue is improved. Therefore, the network has an incentive to increase the capacity to accommodate more users as possible, which also benefits the users in return.

## IV. General Single Link in a Many-User Regime

### A. Uniform Price

Now we have a more general single link problem with $c$ not necessarily equal to 1. The users' utilities are:

$$F_i = w_i \log(1 + k_i x_i) - \frac{1}{nc - \bar{x}} - p x_i, \quad i \in N.$$

Following the same analysis as before, and letting $k_{av}^{-1} := \frac{1}{n}\sum_{j=1}^n \frac{1}{k_j}$, it follows that the unique positive optimal solution is given by:

$$\frac{w_{av} k_{av}^{-1}}{(k_{av}^{-1} + x_{av})^2} - \frac{c + x_{av}}{n^2 (c - x_{av})^3} = 0, \tag{45}$$

$$p = \frac{w_{av}}{k_{av}^{-1} + x_{av}} - \frac{1}{n^2(c - x_{av})^2}, \tag{46}$$

if and only if

$$w_i k_i > \frac{w_{av}}{k_{av}^{-1} + x_{av}}, \quad \forall\ i \in N. \tag{47}$$

It does not seem to be possible to obtain explicit expressions for this case. However, if $w_{av}$ and $k_{av}^{-1}$ converge as $n \to \infty$, then from (45) a positive solution exists for large $n$ if and only if $\lim_{n\to\infty} n^2(c - x_{av})^3 = \alpha$ for some $\alpha > 0$. Thus, $x_{av} \sim c - \alpha^{\frac{1}{3}} n^{-\frac{2}{3}}$, and by substituting this in (45),

$$\alpha = \frac{2c(c + k_{av}^{-1})^2}{w_{av} k_{av}^{-1}}, \tag{48}$$

$$x^*_{av-u}(n) \sim c - \alpha^{\frac{1}{3}} n^{-\frac{2}{3}}, \tag{49}$$

$$d^*_u(n) = \frac{1}{nc - n x^*_{av-u}(n)} \sim \alpha^{-\frac{1}{3}} n^{-\frac{1}{3}}, \tag{50}$$

$$x^*_{i-u}(n) \sim \frac{w_i}{w_{av}}(c + k_{av}^{-1}) - \frac{1}{k_i} - \frac{w_i}{w_{av}}\alpha^{\frac{1}{3}} n^{-\frac{2}{3}}, \tag{51}$$

$$p^*_u(n) \sim \frac{w_{av}}{c + k_{av}^{-1}} + (\frac{2c}{k_{av}^{-1}} - 1)\alpha^{-\frac{2}{3}} n^{-\frac{2}{3}}, \tag{52}$$

$$r^*_u(n) = \frac{p^*_u(n) x^*_{av-u}(n)}{c} \sim \frac{w_{av}}{c + k_{av}^{-1}} - 3\alpha^{-\frac{2}{3}} n^{-\frac{2}{3}}. \tag{53}$$

The necessary and sufficient condition is:

$$w_i k_i > \frac{w_{av}}{c + k_{av}^{-1}} + \frac{2c}{k_{av}^{-1}}\alpha^{-\frac{2}{3}} n^{-\frac{2}{3}}, \quad \forall\ i \in N. \tag{54}$$

### B. Differentiated Prices

Let $z_{av}^{\frac{1}{2}} := \frac{1}{n}\sum_{j=1}^n \sqrt{\frac{w_j}{k_j}}$. Assume $k_{av}^{-1}$ and $z_{av}^{\frac{1}{2}}$ converge as $n \to \infty$. Then the positive solution for large $n$ is:

$$\beta = \frac{2c(c + k_{av}^{-1})^2}{(z_{av}^{\frac{1}{2}})^2}, \tag{55}$$

$$x^*_{av-d}(n) \sim c - \beta^{\frac{1}{3}} n^{-\frac{2}{3}}, \tag{56}$$

$$d^*_d(n) = \frac{1}{nc - n x^*_{av-d}(n)} \sim \beta^{-\frac{1}{3}} n^{-\frac{1}{3}}, \tag{57}$$

$$x^*_{i-d}(n) \sim \frac{\sqrt{w_i/k_i}}{z_{av}^{\frac{1}{2}}}(c + k_{av}^{-1}) - \frac{1}{k_i} - \frac{\sqrt{w_i/k_i}}{z_{av}^{\frac{1}{2}}}\beta^{\frac{1}{3}} n^{-\frac{2}{3}}, \tag{58}$$

$$p^*_{i-d}(n) \sim \frac{z_{av}^{\frac{1}{2}}\sqrt{w_i k_i}}{c + k_{av}^{-1}} + (\frac{2c\sqrt{w_i k_i}}{z_{av}^{\frac{1}{2}}} - 1)\beta^{-\frac{2}{3}} n^{-\frac{2}{3}}, \tag{59}$$

$$r_d^*(n) \sim \frac{w_{av}}{c} - \frac{(z_{av}^{\frac{1}{2}})^2}{c(c+k_{av}^{-1})} - 3\beta^{-\frac{2}{3}} n^{-\frac{2}{3}}, \tag{60}$$

if and only if: $\forall\ i \in N$,

$$\sqrt{w_i k_i} > \frac{z_{av}^{\frac{1}{2}}}{k_{av}^{-1} + x_{av-d}^*(n)} \sim \frac{z_{av}^{\frac{1}{2}}}{c+k_{av}^{-1}} + \frac{2c}{z_{av}^{\frac{1}{2}}} \beta^{-\frac{2}{3}} n^{-\frac{2}{3}}. \tag{61}$$

### C. Comparison of the Two Pricing Schemes

*Theorem 3:* The same conclusion as in Theorem 2 can be reached for the general single link in a many-user regime.

*Proof:* (61) is equivalent to: $w_i k_i > (z_{av}^{\frac{1}{2}})^2/(c + k_{av}^{-1})^2 + [4c/(c+k_{av}^{-1})]\beta^{-\frac{2}{3}} n^{-\frac{2}{3}}$. Furthermore, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{2}\big[(\sqrt{\frac{w_i}{k_j}})^2 + (\sqrt{\frac{w_j}{k_i}})^2\big] \geq \sum_{i=1}^{n}\sum_{j=1}^{n} \sqrt{\frac{w_i}{k_j}}\sqrt{\frac{w_j}{k_i}},$$

or equivalently,

$$w_{av} k_{av}^{-1} \geq \big(z_{av}^{\frac{1}{2}}\big)^2. \tag{62}$$

Thus, $w_{av}/(c + k_{av}^{-1}) \geq (z_{av}^{\frac{1}{2}})^2/[k_{av}^{-1}(c + k_{av}^{-1})] > (z_{av}^{\frac{1}{2}})^2/(c+k_{av}^{-1})^2$, which means (54) leads to (61). ∎

Hereinafter assume that the number of admitted users for UniPri and that for DiffPri are asymptotically the same, as $n \to \infty$. Qualitatively this is actually not a restrictive assumption, because similar results can be obtained even without this simplification.

*Throughput:* As $n \uparrow \infty$, $x_{av-u}^*(n) \uparrow c$ and $x_{av-d}^*(n) \uparrow c$. Thus, $\tilde{x}_{av-u}^* \cong \tilde{x}_{av-d}^*$.

*Congestion Cost:* A byproduct of (62) is $\alpha \leq \beta$. Hence, $\tilde{d}_u^* \geq \tilde{d}_d^*$.

*Individual Flows:* As $n \uparrow \infty$, $x_{i-u}^*(n) \uparrow \frac{w_i}{w_{av}}(c + k_{av}^{-1}) - \frac{1}{k_i}$ and $x_{i-d}^*(n) \uparrow \frac{\sqrt{w_i/k_i}}{z_{av}^{\frac{1}{2}}}(c+k_{av}^{-1}) - \frac{1}{k_i}$. Therefore,

$$\begin{cases} \tilde{x}_{i-u}^* > \tilde{x}_{i-d}^* & \text{if } w_i k_i > \tilde{wk}, \\ \tilde{x}_{i-u}^* = \tilde{x}_{i-d}^* & \text{if } w_i k_i = \tilde{wk}, \\ \tilde{x}_{i-u}^* < \tilde{x}_{i-d}^* & \text{if } w_i k_i < \tilde{wk}, \end{cases} \tag{63}$$

where

$$(w_i k_i)_{\max} \geq \tilde{wk} := \big(w_{av}/z_{av}^{\frac{1}{2}}\big)^2 \geq (w_i k_i)_{\min}. \tag{64}$$

*Prices:* As $n \uparrow \infty$, $p_u^*(n) \downarrow w_{av}/(c+k_{av}^{-1})$ and $p_{i-d}^*(n) \downarrow z_{av}^{\frac{1}{2}}\sqrt{w_i k_i}/(c+k_{av}^{-1})$. Therefore,

$$\begin{cases} \tilde{p}_u^* < \tilde{p}_{i-d}^* & \text{if } w_i k_i > \tilde{wk}, \\ \tilde{p}_u^* = \tilde{p}_{i-d}^* & \text{if } w_i k_i = \tilde{wk}, \\ \tilde{p}_u^* > \tilde{p}_{i-d}^* & \text{if } w_i k_i < \tilde{wk}. \end{cases} \tag{65}$$

*Revenue:* As $n \uparrow \infty$, $r_u^*(n) \uparrow w_{av}/(c+k_{av}^{-1})$ and $r_d^*(n) \uparrow w_{av}/c - (z_{av}^{\frac{1}{2}})^2/[c(c+k_{av}^{-1})]$. By (62), $\tilde{r}_u^* \leq \tilde{r}_d^*$.

*Individual Utilities:* From the above results, we have

$$\begin{cases} \tilde{F}_{i-u}^* > \tilde{F}_{i-d}^* & \text{if } w_i k_i > \tilde{wk}_F, \\ \tilde{F}_{i-u}^* = \tilde{F}_{i-d}^* & \text{if } w_i k_i = \tilde{wk}_F, \\ \tilde{F}_{i-u}^* < \tilde{F}_{i-d}^* & \text{if } w_i k_i < \tilde{wk}_F, \end{cases} \tag{66}$$

where

$$\tilde{wk}_F \geq \tilde{wk}. \tag{67}$$

## V. Concluding Remarks

The contributions of this paper are multifold. First, the existence of a unique Stackelberg/Nash equilibrium is established for a general network modeled as a two-level hierarchical game. Second, the results in [1] are extended to a differentiated price scenario, showing that the revenue per unit bandwidth improves as more users join the network and as a result the service provider has an incentive to increase the capacity in proportion to the number of users, which will in return make the users' utilities better off. The network performance enhances as well in terms of congestion. Last but not the least, one focus of this paper has been to compare differentiated pricing with uniform pricing, and this has led to some conclusive results. With price differentiation, both the service provider and the users with small utility parameters are better off whereas the users with relatively higher utility weight parameters are worse off. We also have smaller congestion cost and a larger number of users admitted for service as a result of price differentiation. Hence, optimum price differentiation leads to a more egalitarian distribution of the resources at fairer prices, which is much intuitive for the design of pricing schemes for the Internet.

For future research, these results can be extended to encompass the linear network model in [2]. Also, they can be extended to more general models which capture competition among multiple service providers, that is, a game with multiple leaders and multiple followers.

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