Network Game with a Probabilistic Description of User Types

Hongxia Shen and Tamer Başar

***Abstract*— This paper presents a hierarchical network game of ISPs and users, and introduces three classes of network games based on information structure: complete information, partially incomplete information, and totally incomplete information. Following the approach developed in [5], the existence and uniqueness of the Nash equilibrium is established for the users' game under complete information (and thus also under partially incomplete information). Furthermore, a sufficient condition for the existence of a unique and stable pure strategy Bayesian equilibrium is obtained for a special two user case under totally incomplete information. The multiple user case with quadratic utility functions is also investigated. Our analysis concludes that incomplete information is a disadvantage for the ISP and the less aggressive users, but is advantageous for the more aggressive users. Also, numerical results indicate that whether the ISP benefits or not from partially incomplete information (users sharing information) compared with totally incomplete information is determined by the number of users.**

***Index Terms*— Noncooperative game, Stackelberg game, Nash equilibrium, complete information, partially incomplete information, totally incomplete information, pure strategy Bayesian equilibrium.**

I. INTRODUCTION

The Internet can be regarded as a competitive market where the Internet Service Providers (ISPs) offer and sell service, and users are the buyers of this service [1], [2]. Thus, the pricing issue may be studied within a game theoretic framework [3], [4]. For this, we can model the problem as a hierarchical Stackelberg (leader-follower) game, for which the ISPs announce prices as leaders and the users respond with flows (usage) as followers. An ISP's maximizing goal is his profit, while a user tries to make his net utility (utility minus payment) as large as possible. Additionally, the competition among the ISPs/users can be captured by the concept of Nash equilibrium. Related discussions on the network game can be found in [5], [6], [7].

In this paper, we study a network with a single ISP and thus focus on the noncooperative game played by the users as well as the Stackelberg game played by the ISP with the users. This problem can be further classfied based on the information structure [3]. One modeling assumption would be to take the utility function of each user as common knowledge for all users and the ISP. We call this the *Complete Information* game. However, in reality complete information is not always the case. Especially for the Internet, there is a large number of users so that the ISP and other users may not have precise knowledge on any specific user's utility function. Consequently, the utility function of each user can be modeled as stochastic; in other words, each user's type is not deterministic, for which only the distribution is known to the players (of course, each user has full information on his own type). If the users are allowed to play the noncooperative game repeatedly, then they can actually figure out each other's real type and just play as in the complete information game. This is called the *Partially Incomplete Information* game. On the contrary, if the users only play the game once and thus do not know each other's type, then it would be the *Totally Incomplete Information* game.

The paper is organized as follows. First, some general assumptions on users' utility functions are made and the optimization problems of users and ISPs are defined for the three classes of information structure. The existence of a unique Nash equilibrium is established for the users' game under complete information. Then the single user case, a special two user case, and the multiple user case with quadratic utility functions are discussed in the next three sections. In particular, for the two user case, a sufficient condition for the existence of a unique stable pure strategy Bayesian equilibrium is achieved for the users' game. Numerical results are also provided in all the three sections. The paper concludes with a discussion of possible extensions.

II. PROBLEM FORMULATION

A. Notation

The setting in this paper assumes a single ISP and multiple users. Let the user set be $\mathcal{I} := \{i : 1 \leq i \leq I\}$ and User i has a flow x_i. Denote the flow vector of all users by $\vec{x} := (x_1, \cdots, x_I)^T$, which is nonnegative and is in some compact convex set Ω. User i's utility function is the sum of two parts: $f_i(x_i)$, which is a strictly increasing, strictly concave nonnegative function of his own flow, and $g(\vec{x})$, a coupled nonincreasing concave function determined by the flows of all users. $g(\vec{x})$ usually models the negative utility due to delay or congestion in the network. Without loss of generality, we assume that it is common to all the users up to scaling. Finally, User i's net utility is $f_i(x_i)+g(\vec{x})-p_i x_i$, where p_i is the price per unit flow charged by the ISP to User i, which must be nonnegative. Denote the price vector by $\vec{p} := (p_1, \cdots, p_I)^T$. Then, the ISP's profit is $\vec{p}^T\vec{x} = \sum_{i\in\mathcal{I}} p_i x_i$.

This work was supported by NSF Grants CCR 00-85917 ITR and ANI-031976.

H.-X. Shen is with the Decision and Control Laboratory, Coordinated Science Laboratory and Department of Electrical and Computer Engineering, University of Illinois, Urbana, Illinois, USA (hshen1@uiuc.edu).

T. Başar is the Fredric G. and Elizabeth H. Nearing Professor, Decision and Control Laboratory, Coordinated Science Laboratory and Department of Electrical and Computer Engineering, University of Illinois, Urbana, Illinois, USA (tbasar@control.csl.uiuc.edu).

B. Optimization Problems

1) Complete Information: In this game, the ISP first announces the price vector $\vec{p}$ and then the users play an I-player noncooperative game adopting the Nash equilibrium solution concept. User i's optimization problem in this game would be:

$$\max_{x_i:x_i\geq 0} f_i(x_i)+g(\vec{x})-p_ix_i, \quad (1)$$

where $x_j,\ j\in\mathcal{I},\ j\neq i$, are held fixed. Denote the Nash equilibrium, if it exists and is unique, by $\vec{x}(\vec{p})=(x_1(\vec{p}),\cdots,x_I(\vec{p}))^T$. Then the ISP plays a Stackelberg game with the users and his optimization problem is:

$$\max_{\vec{p}:\vec{p}\geq\theta} \vec{p}^T\vec{x}(\vec{p})=\max_{\vec{p}:\vec{p}\geq\theta}\sum_{i\in\mathcal{I}} p_ix_i(\vec{p}). \quad (2)$$

Write the optimal price vector as $\vec{p}^*$. Then the optimal solution to the network game is $(\vec{p}^*,\vec{x}(\vec{p}^*))$.

Following the approach in [5], one can show that (1) is equivalent to:

$$\max_{x_i:x_i\geq 0}\{F_{\vec{p}}(\vec{x}):=\sum_{j\in\mathcal{I}} f_j(x_j)+g(\vec{x})-\sum_{j\in\mathcal{J}} p_jx_j\}. \quad (3)$$

Since $f_i(x_i)$, for all $i\in\mathcal{I}$, is strictly concave and $g(\vec{x})$ is concave, (3) admits a unique optimal solution $\vec{x}(\vec{p})$ in Ω, which is also the unique Nash equilibrium for (1) given $\vec{p}$. In this way, we establish the existence and uniqueness of the Nash equilibrium for the users' game under complete information (and thus also under partially incomplete information) for each price vector $\vec{p}$.

2) Partially Incomplete Information: Now the first part of User i's utility is denoted as $f_i^{w_i}(x_i)$, where w_i is a random variable representing his type, with w_i's being independent of each other. Define $\vec{w}:=(w_1,\cdots,w_I)^T$. The second part, $g(\vec{x})$, is deterministic. Given a price vector $\vec{p}$, for each fixed $\vec{w}$, User i's optimization problem is the same as that under complete information:

$$\max_{x_i:x_i\geq 0} f_i^{w_i}(x_i)+g(\vec{x})-p_ix_i. \quad (4)$$

Let $\vec{x}^{\vec{w}}(\vec{p})=(x_1^{\vec{w}}(\vec{p}),\cdots,x_I^{\vec{w}}(\vec{p}))^T$ be the unique Nash equilibrium. Then the ISP's optimization problem becomes:

$$\max_{\vec{p}:\vec{p}\geq\theta}\sum_{i:i\in\mathcal{I}} p_iE_{\vec{w}}[x_i^{\vec{w}}(\vec{p})], \quad (5)$$

for which we denote the optimal price vector by $\vec{p}^*$.

3) Totally Incomplete Information: Since User i does not know others' types, his optimization problem is based on the expectation with respect to $w_{-i}:=\{w_j:j\in\mathcal{I},j\neq i\}$ as follows:

$$\max_{x_i:x_i\geq 0} f_i^{w_i}(x_i)+E_{w_{-i}}[g(\vec{x})]-p_ix_i, \quad (6)$$

for which we denote the pure strategy Bayesian (Nash) equilibrium [3] by $x_i^{w_i}(\vec{p})$ for User i. Assuming that this solution is unique, the ISP's optimization problem is:

$$\max_{\vec{p}:\vec{p}\geq\theta}\sum_{i:i\in\mathcal{I}} p_iE_{w_i}[x_i^{w_i}(\vec{p})], \quad (7)$$

for which we again denote the optimal price vector by $\vec{p}^*$.

III. Single User

We first study the case with a single user, for which there is no distinction between partially incomplete information and totally incomplete information.

A. Complete Information

For the single user case, the notation defined previously can be simplified such that the user's flow is denoted as x and his utility function is $f(x)+g(x)$. Besides, we additionally assume that $f(x)$ and $g(x)$ are twice continuously differentiable for the analysis in this section. Then, from the strict concavity of $f(x)+g(x)$, we know that $f''(x)+g''(x)<0$, or equivalently, $f'(x)+g'(x)$ is strictly decreasing. Without loss of generality, let $\Omega=[0,B]$, for which $f'(x)+g'(x)$ is positive at $x=0$ and equals to 0 at $x=B$. Obviously, the optimal flow must be in Ω. For the ISP side, let p be the price charged per unit flow. Correspondingly, the optimal solutions to (1) given p and to (2) are written as $x(p)$ and p^*, respectively.

Before stating the theorem that captures the optimal solution, we first define the dominance of functions. A function $h_1(x)$ is said to be (strictly) dominated by another function $h_2(x)$ over Ω if $h_1(x)(<)\leq h_2(x)$ for all $x\in\Omega$.

Theorem 1: Assume that $f'(x)+g'(x)$ is strictly dominated by the hyperbola, $h(x)x=$ constant c, which is tangent to it, except at the tangent point. Then there exists a unique optimal solution to the network game, $(p^*,x(p^*))$, such that $x(p^*)\in\Omega$ solves $f'(x)+g'(x)=p^*$ and $(p^*,x(p^*))$ is the tangent point of $f'(x)+g'(x)$ with that hyperbola $h(x)x=p^*x(p^*)$.

Proof: Note that $f(x)+g(x)$ is strictly concave over the compact set Ω. Therefore, given $p\geq 0$, there exists $x\in\Omega$ such that $f'(x)+g'(x)=p$, which is in fact the unique optimal solution $x(p)$ to the user's problem for the given price. If the price is too high, $x(p)=0$. Now along the curve $f'(x)+g'(x)=p$, we know $px=c$ at the tangent point and $px<c$ at any other point since $f'(x)+g'(x)$ is strictly dominated by $h(x)$ except at the tangent point. Therefore, px reaches the maximum at this tangent point. Obviously, this is the unique optimal solution. ∎

B. Incomplete Information: Two-point Distribution

In this subsection, we provide some general analysis for the case when the user's utility is $f^1(x)+g(x)$ w.p. $q_1\in(0,1)$ and $f^2(x)+g(x)$ w.p. $q_2=1-q_1$. The next subsection deals with the continuous distribution case, but in the context of a numerical example.

Now for each p, let $x^1(p)$ and $x^2(p)$ be the optimal solutions to (4) for the two-point distribution case. Then the objective function in (5) becomes $p[q_1x^1(p)+q_2x^2(p)]$, for which the optimal price is p^*. For comparison, let p^{1*} and p^{2*} be the optimal prices under complete information when the user's utility is $f^1(x)+g(x)$ and $f^2(x)+g(x)$, respectively.

Lemma 1: Suppose that ${f^1}'(x)+g'(x)$ and ${f^2}'(x)+g'(x)$ are strictly dominated by the hyperbolae tangent to them, except at the respective tangent points. If $f^{2'}(x)$ is strictly dominated by $f^{1'}(x)$, then $p^{2*}x^2(p^{2*}) < p^{1*}x^1(p^{1*})$; additionally, if $f^{1'}(x) + g'(x)$ and $f^{2'}(x) + g'(x)$ are approximately linear and parallel to each other, then $p^{2*} < p^{1*}$ and $x^2(p^{2*}) < x^1(p^{1*})$.

Proof: The hyperbolae tangent to $f^{1'}(x) + g'(x)$ and $f^{2'}(x) + g'(x)$ are $px = p^{1*}x^1(p^{1*})$ and $px = p^{2*}x^2(p^{2*})$, respectively. Thus, if $f^{2'}(x)$ is dominated by $f^{1'}(x)$, the second hyperbola is also dominated by the first one, whence $p^{2*}x^2(p^{2*}) < p^{1*}x^1(p^{1*})$. Furthermore, if $f^{2'}(x) + g'(x)$ is approximately parallel to $f^{1'}(x) + g'(x)$, then the second problem can be regarded as a version of the first one obtained by appropriate scaling, whence we can deduce the properties given in the algorithm. ∎

Theorem 2: Under the assumptions in Lemma 1, the unique optimal price p^* is achieved at the tangent point of $[q_1 f^{1'}(x) + q_2 f^{2'}(x)] + g'(x)$ with the hyperbola tangent to it, and this unique optimal solution satisfies:

$$p^{2*} < p^* < p^{1*}, \tag{8}$$

$$x^2(p^{2*}) > x^2(p^*), \quad x^1(p^{1*}) < x^1(p^*), \tag{9}$$

$$p^{2*}x^2(p^{2*}) < p^*[q_1 x^1(p^*) + q_2 x^2(p^*)] < p^{1*}x^1(p^{1*}), \tag{10}$$

$$p^*[q_1 x^1(p^*) + q_2 x^2(p^*)] < q_1 p^{1*}x^1(p^{1*}) + q_2 p^{2*}x^2(p^{2*}). \tag{11}$$

Proof: Given any $p \geq 0$, we have $p = f^{1'}(x^1(p)) + g'(x^1(p)) =: h^{1'}(x^1(p))$ and $p = f^{2'}(x^2(p)) + g'(x^2(p)) =: h^{2'}(x^2(p))$. By assumption, $h^{1'}(x) \doteq h^{2'}(x) + a$, which is approximately linear, for some $a > 0$. Thus,

$$\begin{aligned} p &= q_1 h^{1'}(x^1(p)) + q_2 h^{2'}(x^2(p)) \\ &\doteq q_1[h^{2'}(x^1(p)) + a] + q_2 h^{2'}(x^2(p)) \\ &\doteq h^{2'}(q_1 x^1(p) + q_2 x^2(p)) + q_1 a \\ &\doteq q_1 h^{1'}(q_1 x^1(p) + q_2 x^2(p)) + q_2 h^{2'}(q_1 x^1(p) + q_2 x^2(p)). \end{aligned}$$

This proves the first half as a direct result of Theorem 1. Furthermore, since $[q_1 f^{1'}(x) + q_2 f^{2'}(x)] + g'(x)$ runs between and parallel to $h^{1'}(x)$ and $h^{2'}(x)$, (8) (and thus (9)) and (10) follow immediately from Lemma 1.

On the other hand, for any $p \geq 0$, we must have $px^1(p) \leq p^{1*}x^1(p^{1*})$ and $px^2(p) \leq p^{2*}x^2(p^{2*})$, with strict inequality for at least one. Letting $p = p^*$, we obtain (11). ∎

Note that the difference between the two sides of (11) is the profit loss for the ISP due to incomplete information. For the user side, we can see from (8) and (9) that a type 1 user with dominating $f^{1'}(x)$ gets a lower price and thus an increased flow and net utility under incomplete information, while the results are reverse for a type 2 user with dominated $f^{2'}(x)$. Therefore, incomplete information to the ISP is beneficial to the more aggressive users but harms the less aggressive ones.

C. Incomplete Information: Continuous Distribution — A Numerical Example

We adopt quadratic utility functions for our example. As stated in the problem formulation, $f^w(x)$ is a strictly increasing, strictly concave nonnegative function and $g(x)$ is a nonincreasing concave function, both over a compact convex set Ω. Here, let $\Omega = [0, c]$, $f^{w'}(x) = w(1 - c^{-1}x)$, where w is uniformly distributed over $[0, b]$, $g'(x) = -ac^{-1}x$, and a, b, and c are some positive constants.

For the complete information game for which w is known to the ISP, it can be easily calculated from (1) and (2) that the unique optimal solution is: $p^* = w/2$, $x(p^*) = cw/[2(w + a)]$ and $p^*x(p^*) = cw^2/[4(w + a)]$. Then the expected profit for the ISP is $E_w[p^*x(p^*)] = \frac{c}{4b}[\frac{b^2}{2} - ab + a^2 \log \frac{b+a}{a}]$.

Under incomplete information, we can deduce from (4) and (5) that the unique optimal price p^* is the unique solution in $(0, b)$ which satisfies $b - p^* = (2p^* + a)\log \frac{b+a}{p^*+a}$. Then for any w, the user's optimal flow $x^w(p^*)$ is $c(w - p^*)/(w + a)$ if $p^* < w$, and is 0 if $p^* \geq w$. Finally, the expected flow is $E_w[x^w(p^*)] = \frac{c}{b}[b - p^* - (p^* + a)\log \frac{b+a}{p^*+a}]$. The ISP's profit for each w is $p^*x^w(p^*)$ and the expected profit is $p^*E_w[x^w(p^*)]$.

Let a, b, and c all take the value of 1. The results are shown in Fig. 1. The expected profit under complete information is 0.0483. Under incomplete information, the optimal price is $p^* = 0.3176$, the expected flow is 0.1325 and as a result the expected profit is 0.0421. The ISP has a loss of 12.8% due to incomplete information. Also, the figure verifies our previous conclusion that incomplete information makes the ISP charge a lower price from those users with comparatively large values of w, which leads to an increased flow and net utility for them, but does the contrary for those users with comparatively small values of w.

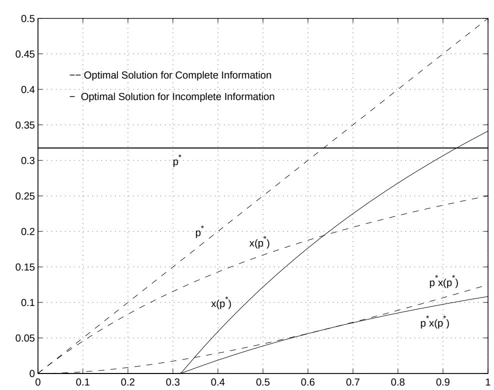

Fig. 1. Optimal solutions for the uniform distribution case with $a = b = c = 1$.

IV. Two Users

A. Nash Equilibrium under Totally Incomplete Information

We have already established the existence of a unique Nash equilibrium for multiple users under complete information (and thus also under partially incomplete information). In the following, we will deduce a sufficient condition for the existence of a unique and stable pure strategy Bayesian equilibrium for a special two user case under totally incomplete information.

Suppose that $f_1(x_1)$ and $f_2(x_2)$ are i.i.d. such that $f_i(x_i)$ is $f^1(x_i)$ w.p. $q_1 \in (0,1)$ and is $f^2(x_i)$ w.p. $q_2 = 1-q_1$, where $f^1(x_i)$ and $f^2(x_i)$ are twice continuously differentiable strictly concave functions, $i=1,2$. Also assume $g(x_1,x_2)$ is deterministic and is twice continuously differentiable and concave over a compact convex set Ω.

Given $\vec{p}=(p_1,p_2)^T$, denote the pure strategy Bayesian equilibrium for (6) by $x_i(\vec{p})=x_i^1(\vec{p})$ if $f_i(x_i)=f^1(x_i)$, and $x_i(\vec{p})=x_i^2(\vec{p})$ if $f_i(x_i)=f^2(x_i)$, $i=1,2$. Then, $x_1^k(\vec{p})$ and $x_2^k(\vec{p})$ solve

$$\max_{x_1^k}\{f^k(x_1^k)+q_1 g(x_1^k,x_2^1(\vec{p}))+q_2 g(x_1^k,x_2^2(\vec{p}))-p_1x_1^k\},$$
$$\max_{x_2^k}\{f^k(x_2^k)+q_1 g(x_1^1(\vec{p}),x_2^k)+q_2 g(x_1^2(\vec{p}),x_2^k)-p_2x_2^k\},$$

respectively, for $k=1,2$. To study the existence of a pure strategy Bayesian equilibrium here, we apply the techniques developed in [8] where the existence, uniqueness and stability of a Nash equilibrium follows from the Banach contraction mapping theorem [9].

From the strict concavity of the users' utility functions, we know that if there exist x_1^1, x_1^2, x_2^1 and x_2^2 such that

$$f^{k\prime}(x_1^k)+q_1\bigtriangledown_1 g(x_1^k,x_2^1)+q_2\bigtriangledown_1 g(x_1^k,x_2^2)-p_1=0, \tag{12}$$

$$f^{k\prime}(x_2^k)+q_1\bigtriangledown_2 g(x_1^1,x_2^k)+q_2\bigtriangledown_2 g(x_1^2,x_2^k)-p_2=0, \tag{13}$$

$k=1,2$, then this yields the values of $x_1^1(\vec{p})$, $x_1^2(\vec{p})$, $x_2^1(\vec{p})$ and $x_2^2(\vec{p})$ for a pure strategy Bayesian equilibrium [10].

In fact, if $(x_1^1,x_1^2)^T$ takes the value of some vector $\vec{u}_0=(u_{10},u_{20})^T$, then $(x_2^1,x_2^2)^T$ can be deduced from (13) as $\vec{v}=(v_1,v_2)^T$. Define this mapping as $\vec{v}=L_2(\vec{u}_0)$. Similarly, from (12) we can obtain $\vec{u}=L_1(\vec{v})$. Thus, $\vec{u}=L_1(L_2(\vec{u}_0))=L_1\circ L_2(\vec{u}_0)=:L_{11}(\vec{u}_0)$. Define $\bigtriangledown L_1(\vec{v})$ as

$$\begin{pmatrix}\bigtriangledown_{11}L_1(\vec{v}) & \bigtriangledown_{12}L_1(\vec{v})\\ \bigtriangledown_{21}L_1(\vec{v}) & \bigtriangledown_{22}L_1(\vec{v})\end{pmatrix}:=\begin{pmatrix}\partial u_1/\partial v_1 & \partial u_2/\partial v_1\\ \partial u_1/\partial v_2 & \partial u_2/\partial v_2\end{pmatrix},$$

and similarly define the gradient matrices of L_2 and L_{11}. Now rewrite (13) for $k=1$ as
$f^{1\prime}(v_1)+q_1\bigtriangledown_2 g(u_{10},v_1)+q_2\bigtriangledown_2 g(u_{20},v_1)-p_2=0$,
and take the derivative with respect to u_{10} on both sides to obtain $\bigtriangledown_{11}L_2(\vec{u}_0)=-[f^{1\prime\prime}(v_1)+q_1\bigtriangledown_{22}^2 g(u_{10},v_1)+q_2\bigtriangledown_{22}^2 g(u_{20},v_1)]^{-1}q_1\bigtriangledown_{21}^2 g(u_{10},v_1)$. Repeat the above step for (12) and (13) with $k=1,2$, respectively. Finally, we have $\bigtriangledown L_{11}(\vec{u}_0)=\bigtriangledown L_2(\vec{u}_0)\cdot\bigtriangledown L_1(\vec{v})$, where

$$\bigtriangledown L_2(\vec{u}_0)=\\ -\begin{pmatrix}q_1 & 0\\ 0 & q_2\end{pmatrix}\begin{pmatrix}\bigtriangledown_{21}^2 g(u_{10},v_1) & \bigtriangledown_{21}^2 g(u_{10},v_2)\\ \bigtriangledown_{21}^2 g(u_{20},v_1) & \bigtriangledown_{21}^2 g(u_{20},v_2)\end{pmatrix}\begin{pmatrix}a_1 & 0\\ 0 & a_2\end{pmatrix},$$
$$\bigtriangledown L_1(\vec{v})=\\ -\begin{pmatrix}q_1 & 0\\ 0 & q_2\end{pmatrix}\begin{pmatrix}\bigtriangledown_{12}^2 g(u_1,v_1) & \bigtriangledown_{12}^2 g(u_2,v_1)\\ \bigtriangledown_{12}^2 g(u_1,v_2) & \bigtriangledown_{12}^2 g(u_2,v_2)\end{pmatrix}\begin{pmatrix}b_1 & 0\\ 0 & b_2\end{pmatrix},$$

with a_1, a_2, b_1 and b_2 equal to

$$\begin{aligned}&[f^{1\prime\prime}(v_1)+q_1\bigtriangledown_{22}^2 g(u_{10},v_1)+q_2\bigtriangledown_{22}^2 g(u_{20},v_1)]^{-1},\\ &[f^{2\prime\prime}(v_2)+q_1\bigtriangledown_{22}^2 g(u_{10},v_2)+q_2\bigtriangledown_{22}^2 g(u_{20},v_2)]^{-1},\\ &[f^{1\prime\prime}(u_1)+q_1\bigtriangledown_{11}^2 g(u_1,v_1)+q_2\bigtriangledown_{11}^2 g(u_1,v_2)]^{-1},\\ &[f^{2\prime\prime}(u_2)+q_1\bigtriangledown_{11}^2 g(u_2,v_1)+q_2\bigtriangledown_{11}^2 g(u_2,v_2)]^{-1},\end{aligned}$$

respectively.

Similarly, starting from some vector $\vec{v}_0$ for $(x_2^1,x_2^2)^T$, we can define $\vec{u}=L_1(\vec{v}_0)$ from (12), $\vec{v}=L_2(\vec{u})$ from (13), and $\vec{v}=L_2(L_1(\vec{v}_0))=L_2\circ L_1(\vec{v}_0)=:L_{22}(\vec{v}_0)$. Then, $\bigtriangledown L_{22}(\vec{v}_0)=\bigtriangledown L_1(\vec{v}_0)\cdot\bigtriangledown L_2(\vec{u})$.

Theorem 3: If
(i) for some $\alpha\in(0,1)$, $\|\bigtriangledown L_{11}(\vec{u}_0)\|\le\alpha<1$ for all feasible $\vec{u}_0$ and $\vec{v}=L_2(\vec{u}_0)$, $\vec{u}=L_1(\vec{v})$, or
(ii) for some $\beta\in(0,1)$, $\|\bigtriangledown L_{22}(\vec{v}_0)\|\le\beta<1$ for all feasible $\vec{v}_0$ and $\vec{u}=L_1(\vec{v}_0)$, $\vec{v}=L_2(\vec{u})$,
then there exists a unique stable pure strategy Bayesian equilibrium for the users' game, i.e. starting from any feasible $\vec{u}_0$ or $\vec{v}_0$, the sequence of $\{\vec{u},\vec{v}\}$ generated by L_1 and L_2 alternatively converges and the limit point is the Bayesian equilibrium.

Proof: (i) implies L_{11} is a contraction mapping and (ii) implies L_{22} is a contraction mapping. In either case, it follows from the Banach contraction mapping theorem that the sequence of $\{\vec{u},\vec{v}\}$ converges, which guarantees the existence, uniqueness and stability of a pure strategy Bayesian equilibrium. ■

B. Quadratic Utility Functions

Assume $f_i^{w_i\prime}(x_i)=w_i(1-c^{-1}x_i)$, $i=1,2$, and $\bigtriangledown g(\vec{x})=-ac^{-1}(x_1+x_2)(1,1)^T$, where $w_1>0$, $w_2>0$, and a and c are positive constants.

1) Complete Information: From (1) and (2), $p_i^*=w_i/2$,

$$x_i(\vec{p}^*)=\frac{c}{2}\cdot\frac{w_1w_2+aw_i-aw_j}{w_1w_2+aw_1+aw_2},\quad i=1,2,\ j\neq i,$$
$$\sum_{i=1}^{2}p_i^*x_i(\vec{p}^*)=\frac{c}{4}\cdot\frac{w_1^2w_2+w_1w_2^2+a(w_1-w_2)^2}{w_1w_2+aw_1+aw_2},$$

subject to $a\le w_1w_2/|w_1-w_2|$. Note that the condition guarantees that $x_i(\vec{p}^*)\ge 0$; if it is not satisfied, one user may have zero flow and thus is not admitted.

2) Partially Incomplete Information: Solving (4) gives

$$x_i^{\vec{w}}(\vec{p})=c\frac{w_1w_2+aw_i-(a+p_i)w_j-ap_i+ap_j}{w_1w_2+aw_1+aw_2}, \tag{14}$$

for $i=1,2$, and $j\neq i$, subject to

$$a(p_i-p_j)+p_iw_j\le w_1w_2+aw_i-aw_j,$$

for $i=1,2$, and $j\neq i$. Substituting (14) in (5), we finally obtain

$$p_i^*=\frac{M_iM_{12}+aM_i^2-aM_1M_2+2aM_0M_{12}}{2(M_1M_2+2aM_0M_1+2aM_0M_2)},\quad i=1,2,$$

where

$$\begin{aligned}&M_{12}:=E_{\vec{w}}[w_1w_2/(w_1w_2+aw_1+aw_2)],\\ &M_i:=E_{\vec{w}}[w_i/(w_1w_2+aw_1+aw_2)],\quad i=1,2,\\ &M_0:=E_{\vec{w}}[1/(w_1w_2+aw_1+aw_2)].\end{aligned}$$

3) Totally Incomplete Information: Suppose that w_1 and w_2 are i.i.d. , with each taking the value of w^1 w.p. $q_1 \in (0,1)$ and w^2 w.p. $q_2 = 1 - q_1$. It is easy to verify that

$$\| \bigtriangledown L_{11}(\vec{u}_0)\| = \frac{aq_1}{w_1 + a} + \frac{aq_2}{w_2 + a} < q_1 + q_2 = 1$$

for any $\vec{u}_0$. Thus, by Theorem 3, there exists a unique and stable pure strategy Bayesian equilibrium for the two users.

To further proceed, notice that the two users are symmetric. Hence, the optimal price vector must satisfy $p_1^* = p_2^* = p^*$. So we can let $p_1 = p_2 = p$ and as a result we must have $x_1^1(p) = x_2^1(p) = x^1(p)$ and $x_1^2(p) = x_2^2(p) = x^2(p)$. After the above simplification, we can solve (6) to obtain

$$x^i(p) = \frac{c[(w^j + a)(w^i - p) + aq_j(w^i - w^j)]}{w^1w^2 + a(w^1 + w^2) + 2a^2 + aq_1w^2 + aq_2w^1},$$

for $i = 1, 2$, and $j \neq i$, subject to

$$(w^j + a)(w^i - p) + aq_j(w^i - w^j) \geq 0,$$

for $i = 1, 2$, and $j \neq i$. Then by (7), it can be deduced that

$$p^* = \frac{w^1w^2 + aq_1w^1 + aq_2w^2}{2(q_1w^2 + q_2w^1 + a)}.$$

4) Numerical Results: The results are shown in Table I to Table III for $a = c = 1$, $w^1 = 2$, $w^2 = 1$ and $q_1 = q_2 = \frac{1}{2}$.

TABLE I
COMPLETE INFORMATION

(w_1, w_2)	(p_1^*, p_2^*)	$(x_1(\vec{p}^*), x_2(\vec{p}^*))$	$\sum_{i=1}^2 p_i^* x_i(\vec{p}^*)$
$(2,2)$	$(1,1)$	$(0.25, 0.25)$	0.5
$(2,1)$	$(1, 0.5)$	$(0.3, 0.1)$	0.35
$(1,2)$	$(0.5, 1)$	$(0.1, 0.3)$	0.35
$(1,1)$	$(0.5, 0.5)$	$(0.1667, 0.1667)$	0.1667
$E_{\vec{w}}[x_i(\vec{p}^*)] = 0.2042$,	$E_{\vec{w}}[\sum_{i=1}^2 p_i^* x_i(\vec{p}^*)] = 0.3417$		

TABLE II
PARTIALLY INCOMPLETE INFORMATION

(w_1, w_2)	(p_1^*, p_2^*)	$(x_1^{\vec{w}}(\vec{p}^*), x_2^{\vec{w}}(\vec{p}^*))$	$\sum_{i=1}^2 p_i^* x_i^{\vec{w}}(\vec{p}^*)$
$(2,2)$	$(0.3, 0.3)$	$(0.4250, 0.4250)$	0.2550
$(2,1)$	$(0.3, 0.3)$	$(0.5400, 0.0800)$	0.1860
$(1,2)$	$(0.3, 0.3)$	$(0.0800, 0.5400)$	0.1860
$(1,1)$	$(0.3, 0.3)$	$(0.2333, 0.2333)$	0.1400
$E_{\vec{w}}[x_i^{\vec{w}}(\vec{p}^*)] = 0.3196$,	$E_{\vec{w}}[\sum_{i=1}^2 p_i^* x_i^{\vec{w}}(\vec{p}^*)] = 0.1918$		

TABLE III
TOTALLY INCOMPLETE INFORMATION

(w_1, w_2)	(p^*, p^*)	$(x_1^{w_1}(p^*), x_2^{w_2}(p^*))$	$\sum_{i=1}^2 p^* x_i^{w_i}(p^*)$
$(2,2)$	$(0.7, 0.7)$	$(0.3647, 0.3647)$	0.5106
$(2,1)$	$(0.7, 0.7)$	$(0.3647, 0.0471)$	0.2882
$(1,2)$	$(0.7, 0.7)$	$(0.0471, 0.3647)$	0.2882
$(1,1)$	$(0.7, 0.7)$	$(0.0471, 0.0471)$	0.0659
$E_{w_i}[x_i^{w_i}(p^*)] = 0.2059$,	$\sum_{i=1}^2 p^* E_{w_i}[x_i^{w_i}(p^*)] = 0.2882$		

We can see that the ISP makes the most profit under complete information, while the worst case for the ISP is the partially incomplete information game because of information asymmetry. Evidently, the users have more information than the ISP has in the latter case and thus can take advantage of that to claim more network services.

V. MULTIPLE USERS

Now we study the multiple user case, assuming quadratic utility functions, i. e. , $f_i'(x_i) = w_i(1 - x_i)$ and $\frac{\partial g(\vec{x})}{\partial x_i} = -\sum_{j=1}^I x_j$, $1 \leq i \leq I$.

A. Complete Information

The complete information network game under uniform and differentiated pricing has been studied in [7], with a user's utility function being a logarithmic function minus the inverse of the total flow. The complete information case here is actually an extension of differentiated pricing in that previous work to the problem with quadratic utility functions. Following similar analysis, we can obtain from (1) and (2) the solution to the optimization problems as follows: order the users in the nonincreasing order of w_i's; find the largest number n, $1 \leq n \leq I$, such that $w_i(1 + \sum_{j=1}^n \frac{1}{w_j}) > n$, $1 \leq i \leq n$; [1] then, $p_i^* = \frac{w_i}{2}$ and $x_i(\vec{p}^*) = \frac{1}{2} - \frac{n}{2w_i}(1 + \sum_{j=1}^n \frac{1}{w_j})^{-1}$, $1 \leq i \leq n$, while $p_i^* \geq w_i - \sum_{j=1}^n x_j(\vec{p}^*)$ such that $x_i(\vec{p}^*) = 0$, $n < i \leq I$.

B. Partially Incomplete Information

Hereinafter, assume that w_i's are i.i.d.. Given the ISP's uniform price p and for each fixed $\vec{w}$, the Nash equilibrium can be obtained from (4) similarly as that in [7] under uniform pricing, and it comes out to be: order the users in the nonincreasing order of w_i's; find the largest number n, $0 \leq n \leq I$, such that $w_i(1 + \sum_{j=1}^n \frac{1}{w_j}) > n + p$, $1 \leq i \leq n$; [2] then, $\bar{x}^{\vec{w}}(p) = (n - p\sum_{j=1}^n \frac{1}{w_j})/(1 + \sum_{j=1}^n \frac{1}{w_j})$, $x_i^{\vec{w}}(p) = 1 - \frac{1}{w_i}(p + \bar{x}^{\vec{w}}(p))$, $1 \leq i \leq n$, while $x_i^{\vec{w}}(p) = 0$, $n < i \leq I$. Finally, the optimal price p^* is the solution to the ISP's problem (5) and is determined by the number of users as well as the distribution of w_i's.

C. Totally Incomplete Information

Given p, all the users must have the same strategy: $x_i^{w_i}(p) = x^{w_i}(p)$, $1 \leq i \leq I$. We study here the case with four user types, i.e., $w_i = w^t$ with probability $q_t > 0$, $t \in \{1, 2, 3, 4\}$, where $w^4 > w^3 > w^2 > w^1$ and $\sum_{t=1}^4 q_t = 1$. Given the ISP's price p, denote the corresponding flow to type t by $x^t(p)$. Then, the users' optimization problem (6) can be equivalently written as:

$$w^t(1 - x^t(p)) - x^t(p) - (I - 1)E_t[x^t(p)] = p, \; x^t(p) > 0,$$
$$w^t - (I - 1)E_t[x^t(p)] \leq p, \; x^t(p) = 0,$$

where $E_t[x^t(p)] = \sum_{t=1}^4 q_t x^t(p)$. Then it can be deduced that there exist five possible cases depending on the price.

[1] Since w_i's are nonincreasing, the existence and uniqueness of such n can be easily verified.

[2] A unique such n exists here also, for the same reason as above.

1) Case 1: $x^t(p) = 0$, $4 \geq t \geq 1$. The necessary and sufficient condition for this case is: $p \geq w^4$.

2) Cases 2, 3 and 4: For $k = 4, 3, 2$, respectively, if
$w^k - (I-1)\sum_{t=k+1}^{4} q_t(w^t - w^k)(1+w^t)^{-1}$
$> p \geq w^{k-1} - (I-1)\sum_{t=k}^{4} q_t(w^t - w^{k-1})(1+w^t)^{-1}$,
then

$x^t(p) > 0$, $4 \geq t \geq k$, and $x^t(p) = 0$, $k-1 \geq t \geq 1$.

The expected flow of each user is

$$E_t[x^t(p)] = \frac{\sum_{t=k}^{4} q_t(w^t - p)(1+w^t)^{-1}}{1 + (I-1)\sum_{t=k}^{4} q_t(1+w^t)^{-1}}.$$

3) Case 5: In this case, flows are positive for all types under the condition

$$p < w^1 - (I-1)\sum_{t=2}^{4} q_t(w^t - w^1)(1+w^t)^{-1}.$$

The expected flow of each user is

$$E_t[x^t(p)] = \frac{\sum_{t=1}^{4} q_t(w^t - p)(1+w^t)^{-1}}{1 + (I-1)\sum_{t=1}^{4} q_t(1+w^t)^{-1}}.$$

Finally, according to the ISP's optimization problem (7), the optimal price p^* is the one that maximizes $pE_t[x^t(p)]$.

D. Numerical Results

Assume $w^t = t$ w.p. $q^t = \frac{1}{4}$, $t \in \{1, 2, 3, 4\}$. We compute the results for an example with $I = 40$ users and 10 users for each type. The users are in the nondecreasing order of w_i's. The unique optimal solutions for the three games, respectively, are shown in Table IV (the capital letters in the first row stand for the games with corresponding initials).

For the users, again we can see that the comparatively high types benefit from incomplete information. For the ISP, complete information is still the most ideal case for him, while partially incomplete information leads to a higher profit than the totally incomplete information case does, which is in contrast with our previous conclusions for the two user case.

TABLE IV

NUMERICAL EXAMPLE WITH 40 USERS

	C		P		T	
i, w_i	p_i^*	$x_i(\vec{p}^*)$	p^*	$x_i^{\vec{w}}(p^*)$	p^*	$x^t(p^*)$
1–10, 4	2	0.1341	1.99	0.1436	2	0.1356
11–20, 3	1.5	0.0122	1.99	0	2	0
21–30, 2	≥ 0.5366	0	1.99	0	2	0
31–40, 1	≥ 0	0	1.99	0	2	0
n	20		10		10	
$\sum x_j$	1.4634		1.4357		1.3559	
$\sum p_j x_j$	2.8659		2.8571		2.7119	

Actually, numerical computation shows that under partially incomplete information, if the distribution of w_i's remains unchanged, as the number of users increases, the optimal price for the ISP gets higher, with a higher expected profit as well. For instance, for $I = 1$, 10, 20, 30 and 40, respectively, the optimal price is 1.41, 1.72, 1.84, 1.96 and 1.99, and the expected profit is 0.3920, 1.7036, 2.2128, 2.5557 and 2.8109. This tells us that if the number of users is small, partially incomplete information may hurt the ISP by information asymmetry, while as the number of users gets large, the users' sharing of information can eventually benefit the ISP.

VI. DISCUSSION AND EXTENSIONS

This paper formulates a game theoretical framework for strategic pricing in networks with complete information, partially incomplete information and totally incomplete information. Analyses and numerical results imply that incomplete information decreases the payoff to the ISP and to the less aggressive users and increases the payoff to the more aggressive users. On the other hand, the comparison of the partially incomplete information game and the totally incomplete information game is more complicated, involving consideration of the number of users. When there is a small number of users, the ISP is in favor of totally incomplete information; while as the number of users gets large, the users' sharing of information eventually benefits the ISP in terms of improved profit.

Analyses for the multiple user case and numerical examples in this paper are mainly based on the quadratic utility functions. More general utility functions shall be considered as well in our future work. Also, extensions to general networks with multiple ISPs are currently under study.

REFERENCES

[1] C. Courcoubetis and R. Weber, *Pricing Communication Networks: Economics, Technology and Modelling.* Wiley, England, 2003.

[2] A. Mas-Colell, M. D. Whinston and J. R. Green, *Microeconomic Theory.* Oxford University Press, New York, 1995.

[3] D. Fudenberg and J. Tirole, *Game Theory.* MIT, 1991.

[4] T. Başar and G. J. Olsder, *Dynamic Noncooperative Game Theory.* SIAM Series in Classics in Applied Mathematics, Philadelphia, PA, 1999.

[5] T. Başar and R. Srikant, "Revenue-Maximizing Pricing and Capacity Expansion in a Many-Users Regime," *Proc. IEEE Infocom 2002*, pp. 1556–1563, New York, New York, June 23–27, 2002.

[6] T. Başar and R. Srikant, "A Stackelberg Network Game with a Large Number of Followers," *J. Optimization Theory and Applications*, 115(3): 479–490, Dec. 2002.

[7] H.-X. Shen and T. Başar, "Differentiated Internet Pricing Using a Hierarchical Network Game Model," *Proc. IEEE ACC 2004*, pp. 2322–2327, Boston, MA, 2004.

[8] S. Li and T. Başar, "Distributed Algorithms for the Computation of Noncooperative Equilibria," *Automatica*, 23, pp. 523–533, 1987.

[9] D. G. Luenberger, *Optimization by Vector Space Methods.* Wiley, New York, 1969.

[10] D. P. Bertsekas, *Nonlinear Programming: Second Edition.* Athena Scientific, 1999.