Network Game with a Probabilistic Description of User Types

Hongxia Shen and Tamer Başar

Coordinated Science Laboratory

University of Illinois at Urbana-Champaign

hshen1, tbasar@control.csl.uiuc.edu

CDC 2004, Atlantis, Bahamas

December 17, 2004

Outline

- Previous Work
- Problem Formulation
  - Complete Information
  - Partially Incomplete Information
  - Totally Incomplete Information
- Single User
- Two Users
- Multiple Users
- Conclusions and Extensions

Previous Work

- Başar and Srikant (2002a, 2002b)

  Hierarchical Stackelberg Network Game Model

  Solution under Uniform Pricing

  Asymptotic Behavior Analysis

- Shen and Başar (2004)

  Solution under Differentiated Pricing

  Comparison with Uniform Pricing and Asymptotic Behavior Analysis

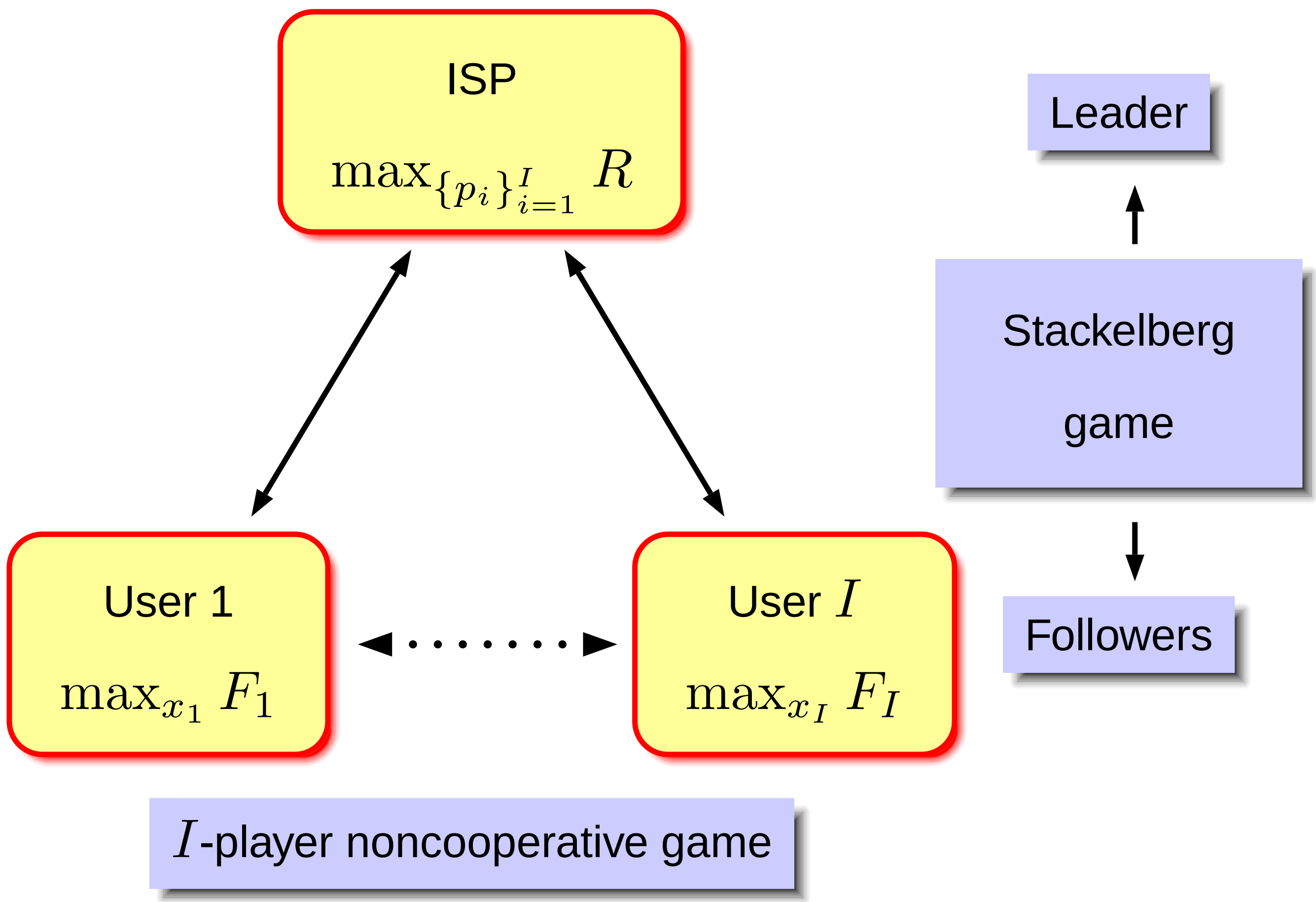
---

T. Başar and R. Srikant (2002a), "Revenue-maximizing pricing and capacity expansion in a many-users regime," *Proc. IEEE INFOCOM 2002,* pp. 1556–1563.

T. Başar and R. Srikant (2002b), "A Stackelberg network game with a large number of followers," *J. Optimization Theory and Applications,* 115(3): 479–490.

H.-X. Shen and T. Başar (2004), "Differentiated Internet pricing using a hierarchical network game model," *Proc. IEEE ACC 2004,* pp. 2322–2327.

Two-Level Hierarchical Network Game



Problem Formulation

- Single Internet Service Provider (ISP)
- Set of users: $\mathcal{I} = \{i : 1 \leq i \leq I\}$
- Flow vector: $\vec{x} = (x_1, \cdots, x_I)^T \in \Omega$
- Price vector: $\vec{p} = (p_1, \cdots, p_I)^T$
- User i's net utility: $f_i(x_i) + g(\vec{x}) - p_i x_i$
- ISP's profit: $\vec{p}^T \vec{x}$
- Classification of games based on the information structure:
  1. Complete information
  2. Partially incomplete information
  3. Totally incomplete information

Complete Information

- The utility function of each user is common knowledge for all the users and the ISP.
- Given $\vec{p}$, a Nash equilibrium is $\vec{x}(\vec{p}) = (x_1(\vec{p}), \cdots, x_I(\vec{p}))^T$, such that for User i,
$$x_i(\vec{p}) = \arg\max_{x_i} f_i(x_i) + g(x_i, x_{-i}(\vec{p})) - p_i x_i,$$
where $x_{-i} = \{x_j\}_{j \neq i}$.
- There exists a unique Nash equilibrium. See Başar and Srikant (2002a).
- The optimal price vector for the ISP is:
$$\vec{p}^* = \arg\max_{\vec{p}:\vec{p} \geq \theta} \sum_{i \in \mathcal{I}} p_i x_i(\vec{p}).$$

T. Başar and R. Srikant (2002a), “Revenue-maximizing pricing and capacity expansion in a many-users regime,” *Proc. IEEE INFOCOM 2002,* pp. 1556–1563.

Partially Incomplete Information

- The utility function of each user is common knowledge for all the users, but not for the ISP.
- The distribution of the users' types, $\vec{w} = (w_1, \cdots, w_I)^T$, that determine f_i's (g is deterministic), is known to the ISP.
- Given $\vec{p}$, for each fixed $\vec{w}$, the unique Nash equilibrium is $\vec{x}^{\vec{w}}(\vec{p}) = (x_1^{\vec{w}}(\vec{p}), \cdots, x_I^{\vec{w}}(\vec{p}))^T$, such that for User i,

$$x_i^{\vec{w}}(\vec{p}) = \arg\max_{x_i} f_i^{w_i}(x_i) + g(x_i, x_{-i}^{\vec{w}}(\vec{p})) - p_i x_i.$$

- The optimal price vector for the ISP is:

$$\vec{p}^* = \arg\max_{\vec{p}:\vec{p}\geq\theta} \sum_{i\in\mathcal{I}} p_i E_{\vec{w}}[x_i^{\vec{w}}(\vec{p})].$$

Totally Incomplete Information

- User i has private information w_i and knows the distribution of $w_{-i} = \{w_j\}_{j \neq i}$.
- The distribution of $\vec{w} = (w_1, \cdots, w_I)^T$ is known to the ISP.
- Given $\vec{p}$, the pure strategy Bayesian equilibrium, if it exists, is $x_i^{w_i}(\vec{p})$ for User i, such that

$$x_i^{w_i}(\vec{p}) = \arg\max_{x_i} f_i^{w_i}(x_i) + E_{w_{-i}}[g(x_i, x_{-i}^{w_{-i}}(\vec{p}))] - p_i x_i.$$

- The optimal price vector for the ISP is:

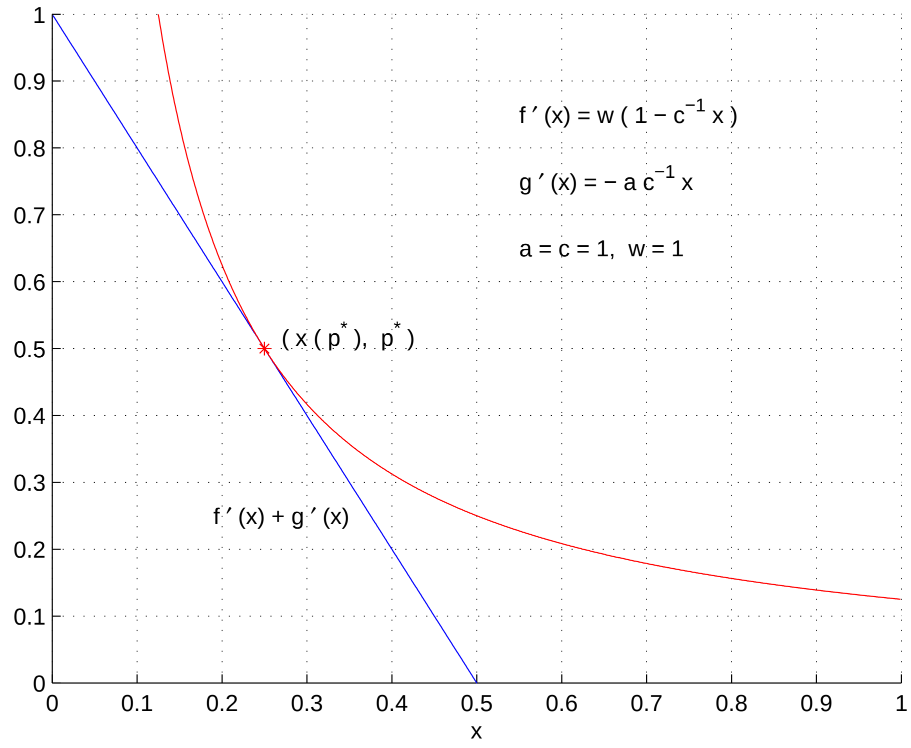
$$\vec{p}^* = \arg\max_{\vec{p}:\vec{p} \geq \theta} \sum_{i \in \mathcal{I}} p_i E_{\vec{w}}[x_i^{w_i}(\vec{p})].$$

Single User

- No Classification for Incomplete Information
- Complete Information
- Incomplete Information: Two-point Distribution
- Incomplete Information: Continuous Distribution
  — A Numerical Example
- Comparison of Complete Information and Incomplete Information

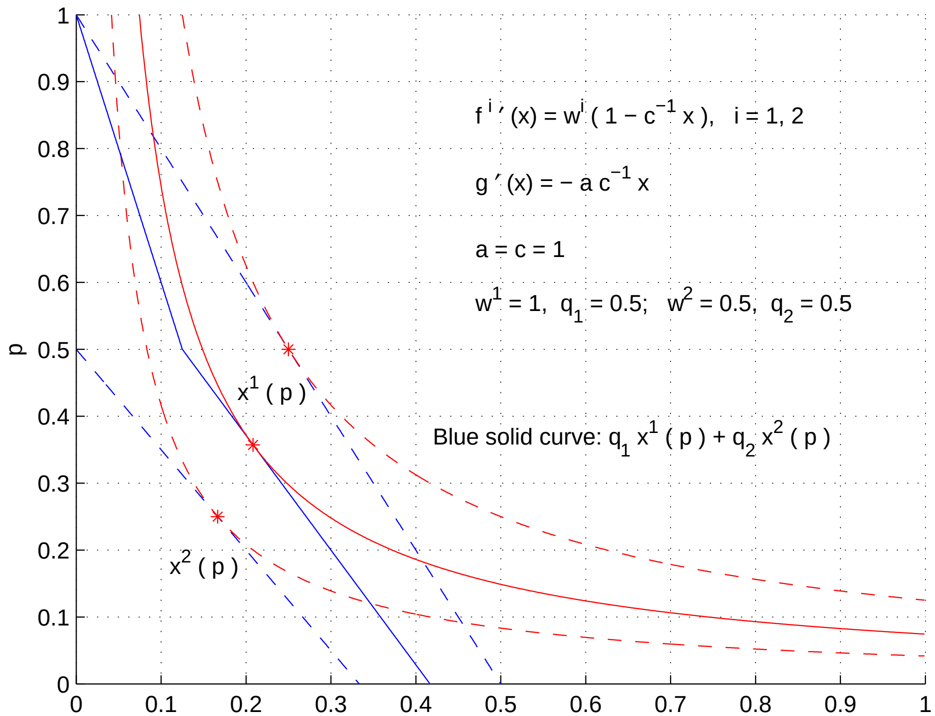
Complete Information

- User: $\max_x f(x) + g(x) - px => p = f'(x(p)) + g'(x(p))$
- ISP: $\max_p px(p)$
- Graphical illustration of the solution



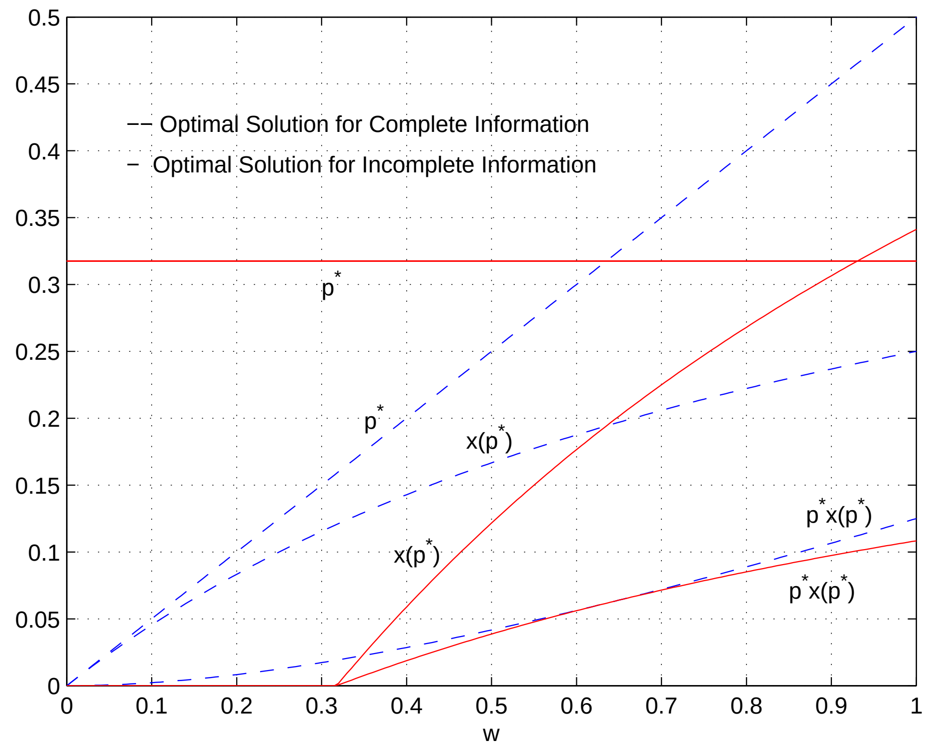
Incomplete Information — Two-point Distribution

- User: $x^i(p) = \arg\max_x f^i(x) + g(x) - px$ w.p. $q_i, \; i = 1, 2$
- ISP: $p^* = \arg\max_p p[q_1 x^1(p) + q_2 x^2(p)]$
- Complete information: $p^{i*} = \arg\max_p p x^i(p), \; i = 1, 2$
- Graphical illustration of the solution and comparison



Incomplete Information — Uniform Distribution

- $f^{w\prime}(x) = w(1 - c^{-1}x),\ g'(x) = -ac^{-1}x$
- w uniformly distributed over $[0, b]$
- Numerical example: $a = b = c = 1$
- Profit loss of 12.8% for the ISP due to incomplete information



Two Users

- Complete Information and Partially Incomplete Information
  — Unique Nash Equilibrium
- Totally Incomplete Information
  — Pure Strategy Bayesian Equilibrium
- Quadratic Utility Functions
  - Complete Information
  - Partially Incomplete Information
  - Totally Incomplete Information
  - Numerical Results
- Comparison of the Three Classes of Games

Pure Strategy Bayesian Equilibrium

- ➢ Special case: independent users $i = 1, 2,\ f_i(x_i) = f^k(x_i)$ w.p. q_k, $k = 1, 2$
- ➢ Pure strategy Bayesian equilibrium: $x_1^k(\vec{p})$ and $x_2^k(\vec{p}),\ k = 1, 2$, solve
$\max_{x_1^k} \left\{ f^k(x_1^k) + q_1 g(x_1^k, x_2^1(\vec{p})) + q_2 g(x_1^k, x_2^2(\vec{p})) - p_1 x_1^k \right\}$,
$\max_{x_2^k} \left\{ f^k(x_2^k) + q_1 g(x_1^1(\vec{p}), x_2^k) + q_2 g(x_1^2(\vec{p}), x_2^k) - p_2 x_2^k \right\}$
- ➢ Sufficient condition for the existence, uniqueness and stability
  - ❶ apply the techniques in Li and Başar (1987)
  - ❷ convergence by the Banach contraction mapping theorem

S. Li and T. Başar (1987), "Distributed algorithms for the computation of noncooperative equilibria," *Automatica,* 23: 523–533.

Quadratic Utility Functions

- $f_i^{w_i\prime}(x_i) = w_i(1 - c^{-1}x_i)$, $i = 1, 2$
  $\bigtriangledown g(\vec{x}) = -ac^{-1}(x_1 + x_2)(1, 1)^T$
  independent users: $i = 1, 2,\ w_i = w^k$ w.p. q_k, $k = 1, 2$
- Analytical results for the three types of games
- Numerical results:
  $a = c = 1,\ w^1 = 2,\ w^2 = 1,\ q_1 = q_2 = 0.5$

Numerical Results and Comparison of Games

(w_1, w_2)	Game	Optimal Price Vector	Optimal Flow Vector	ISP's Profit
$(2,2)$	C	$(1,1)$	$(0.25, 0.25)$	0.5
	P	$(0.3, 0.3)$	$(0.4250, 0.4250)$	0.2550
	T	$(0.7, 0.7)$	$(0.3647, 0.3647)$	0.5106
$(2,1)$	C	$(1, 0.5)$	$(0.3, 0.1)$	0.35
	P	$(0.3, 0.3)$	$(0.5400, 0.0800)$	0.1860
	T	$(0.7, 0.7)$	$(0.3647, 0.0471)$	0.2882
$(1,2)$	C	$(0.5, 1)$	$(0.1, 0.3)$	0.35
	P	$(0.3, 0.3)$	$(0.0800, 0.5400)$	0.1860
	T	$(0.7, 0.7)$	$(0.0471, 0.3647)$	0.2882
$(1,1)$	C	$(0.5, 0.5)$	$(0.1667, 0.1667)$	0.1667
	P	$(0.3, 0.3)$	$(0.2333, 0.2333)$	0.1400
	T	$(0.7, 0.7)$	$(0.0471, 0.0471)$	0.0659

Expected Profit: C — 0.3417 P — 0.1918 T — 0.2882

Multiple Users

- Quadratic Utility Functions
  - Complete Information
  - Partially Incomplete Information
  - Totally Incomplete Information
  - Numerical Results
- Comparison of the Three Classes of Games

Quadratic Utility Functions

- $f_i^{w_i\prime}(x_i) = w_i(1 - x_i)$, $\frac{\partial g(\vec{x})}{\partial x_i} = -\sum_{j=1}^{I} x_j$
  Independent users: $1 \leq i \leq I$
- Complete Information: Nash equilibrium — extension of differentiated pricing in Shen and Başar (2004)
- Partially Incomplete Information: Nash equilibrium — extension of uniform pricing in Shen and Başar (2004)
- Totally Incomplete Information: analytical results for the special case with four user types

---

H.-X. Shen and T. Başar (2004), "Differentiated Internet pricing using a hierarchical network game model," *Proc. IEEE ACC 2004,* pp. 2322–2327.

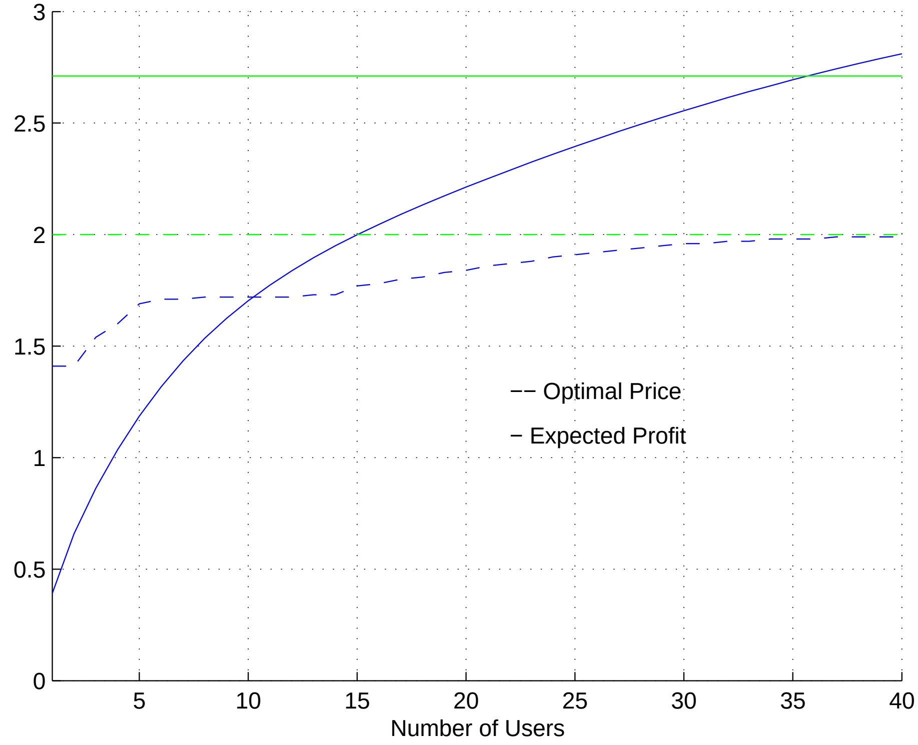
Numerical Results and Comparison of Games

➢ Numerical results:
$w^t = t$ w.p. $q_t = \frac{1}{4}, t \in \{1, 2, 3, 4\}; I = 40, 10$ for each type

	C		P		T	
i, w_i	p_i^*	$x_i(\vec{p}^*)$	p^*	$x_i^{\vec{w}}(p^*)$	p^*	$x^t(p^*)$
1–10, 4	2	0.1341	1.99	0.1436	2	0.1356
11–20, 3	1.5	0.0122	1.99	0	2	0
21–30, 2	≥ 0.5366	0	1.99	0	2	0
31–40, 1	≥ 0	0	1.99	0	2	0
n	20		10		10	
$\sum x_j$	1.4634		1.4357		1.3559	
$\sum p_j x_j$	2.8659		2.8571		2.7119	

Partially? Totally?

➢ Partially Incomplete Information vs Totally Incomplete Information



Conclusions

- ➢ Formulation of the three classes of games
- ➢ Complete Information vs Incomplete Information:

  Complete Information — ISP and less aggressive users

  Incomplete Information — more aggressive users
- ➢ Partially Incomplete Information vs Totally Incomplete Information for the ISP:

  Partially Incomplete Information — with a large number of users

  Totally Incomplete Information — with a small number of users

Extensions

- ➢ General utility functions; Multiple ISPs; . . .

________ End of the Talk ________