Hierarchical Network Games with Various Types of Public and Private Information

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Abstract— **We consider, in this paper, a class of hierarchical network games where there is a single service provider (leader, in a Stackelberg game framework) and multiple users (followers) which could be of different types. Depending on whether the type of a particular user is private information (only to that user), or public information (shared with all users as well as the service provider), or whether we have the intermediate case where this is common (shared) information among the users but not shared with the service provider, one can introduce and study the equilibria of different types of games, covering the entire gamut from complete information to incomplete information games. We undertake such a study in this paper, with general utility functions for the players and general distributions for user characteristics. We compare the performances of the leader and the followers under the different scenarios, and also study the asymptotic case as the user population grows. The study for the many-followers regime provides useful insight for communication network applications.**

Index Terms— **Stackelberg game, private information, public information, Nash equilbrium, Bayesian equilibrium, Wardrop equilibrium.**

I. INTRODUCTION

In recent years, there has developed much interest in the design of optimal congestion control schemes for communication networks. Some previous works study flow control problems (for example, [1], [2]), some focus on routing ([3], [4]), and others examine combined flow control and routing ([5]). While many of these works apply game theory to network optimization, most of them aim at achieving optimal aggregate efficiency for the whole system. [6] and [7] take a different perspective and consider optimizing profits of network service providers. In [6], a hierarchical Stackelberg game model has been introduced to analyze the flow control problem on a single link network, and the work has been extended to a linear network in [7].

In this paper, we study this hierarchical Stackelberg network game model played by a monopolistic service provider and multiple users, with the extension to various types of public and private information. Now, suppose that we have stochastic users, whose types are random variables with certain distributions. In the classical version of Stackelberg game as modeled in [6], it is assumed that the true type of each user is publicly known to the service provider as well as to all the users. This is called a *complete information* game here. We have here two other classes of games: if the true types of users are common shared information among users themselves, but are not disclosed to the service provider, the game is called a *partially incomplete information* game; on the other hand, if each user's true type is private information to him and is not shared with the others, we have a *totally incomplete information* game. Precise problem formulations for these three classes of games will be given in the next section.

Interaction of product suppliers (service providers) and buyers (users) with stochastic types under various types of public and private information has been extensively studied in economics, especially in the context of auction theory and mechanism design (for example, see [8], [9]). However, in the congestion control literature for communication networks, as previously mentioned, it is generally assumed that user types (utility preference or weight parameters) are fixed, though sometimes they may not be known to the network or to the service providers. Also, there lacks a comparative study of complete and incomplete information. In [10], we made an effort in this direction by considering the three classes of network games. Some analytical and numerical results had been obtained, mainly based on special quadratic utility functions and discrete distributions of user types.

This paper is a follow-up to [10], with the extension to general utility functions for the players and general distributions for user characteristics. We first provide precise problem formulations for the three classes of games. Then, we prove the existence and uniqueness of Nash equilibrium, as well as of Bayesian equilibrium. The next section deals with the two-user case, and the study indicates that how the users behave comparatively under partially incomplete information or under totally incomplete information depends on the users' true types. A thorough comparison of the three classes of games, especially from the service provider's perspective, proves to be hard for this case. Accordingly, we turn to the asymptotic case with a large number of users. The analysis shows that the distinction between the partially incomplete information game and the totally incomplete information game vanishes as the user population increases. Thus, the service provider's game preference becomes clear in this case. This insight is especially useful for communication networks with a large population of users. The paper ends with concluding remarks and discussion on future work.

This work was supported in part by NSF Grant ANI-031976.

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II. Problem Formulation

In the Stackelberg game model, the monopolistic service provider announces prices for the service he provides as the leader, and the n users respond with their choices of usage (for communication networks, flows). The service provider's payoff (revenue) is the total payment collected from all the users, and a user's payoff (net utility) is what he gets from using the service deducted by his payment to the service provider.

In this paper, we model the users' utility functions as follows. For User i, $i \in N := \{1, \cdots, n\}$, his net utility consists of three additive terms. The first term is a nonnegative function $f(x_i; w_i)$, modeling the benefit from his own usage, x_i, and f is strictly increasing and strictly concave in x_i [1]. Here, w_i is a one-dimensional parameter representing the user's type. We assume that $f(x; w)$ and $\partial f(x; w)/\partial x$ are strictly increasing in w, which means that the higher is a user's type, the more he benefits from a certain amount of usage increase. The second term is a common negative part $-g(x_1, \cdots, x_n)$, depending on the usage of all the users, which models a user's loss in utility from the competition with the other users for the provided service. We assume that g is convex and symmetric for all the users, i.e., any two arguments x_i and x_j, $i, j \in N$, of $g(x_1, \cdots, x_n)$ can be exchanged without changing the value of the function. Also, for any $i \in N$, $g(x_1, \cdots, x_n)$ and $\partial g(x_1, \cdots, x_n)/\partial x_j$, $j \in N$ and $j \neq i$, are increasing in x_i, which means that a user's usage increase leads to deterioration of the network performance and makes it more sensitive to congestion. The third part of User i's utility is the negative of his payment, $p_i x_i$, where p_i is the price per unit usage charged to him by the service provider.

Now for stochastic users, we assume that all the w_i's are independently and identically distributed, with the common distribution publicly known to all the players. However, whether the true types are known to the players depends on the classes of games, which can be formulated as follows.

A. Complete Information

In the complete information game, all the users' true types, $\mathbf{w} := \{w_i\}_{i \in N}$, are known to all the players. The service provider first announces the prices, $\mathbf{p} := \{p_i\}_{i \in N}$, as the leader, and then the users play a noncooperative game, whose solution, if exists, is a Nash equilibrium. In this Nash equilibrium, User i's usage, $i \in N$, is

$$x_i^c(\mathbf{w}; \mathbf{p}) = \arg\max_{x_i} \{f(x_i; w_i) - g(x_i, \mathbf{x_{-i}}^c(\mathbf{w}; \mathbf{p})) - p_i x_i\}, \tag{1}$$

where $\mathbf{x_{-i}} := \{x_j\}_{j \neq i, j \in N}$. Note that g in (1) should be

$$g(x_1^c(\mathbf{w}; \mathbf{p}), \cdots, x_{i-1}^c(\mathbf{w}; \mathbf{p}), x_i, x_{i+1}^c(\mathbf{w}; \mathbf{p}), \cdots, x_n^c(\mathbf{w}; \mathbf{p})),$$

and we can write it as in (1) because of its symmetry for all the users. Then, for the Stackelberg game, the optimal price vector for the service provider is

$$\mathbf{p}^c(\mathbf{w}) = \arg\max_{\mathbf{p}} \sum_{i \in N} p_i x_i^c(\mathbf{w}; \mathbf{p}). \tag{2}$$

B. Partially Incomplete Information

In the partially incomplete information game, each user's true type is known to the other users, but not to the service provider. In this case, all the users look like the same to the service provider, because (i) all the users' types are independently and identically distributed, and (ii) all the users are symmetric in g and thus in the utility functions. Therefore, the service provider charges a uniform unit price p to all the users. Then the Nash equilibrium for the users' game, if exists, satisfies

$$x_i^p(\mathbf{w}; p) = \arg\max_{x_i} \{f(x_i; w_i) - g(x_i, \mathbf{x_{-i}}^p(\mathbf{w}; p)) - p x_i\}, \tag{3}$$

for $i \in N$. For the Stackelberg game, the optimal price for the service provider is

$$p^p = \arg\max_p \; p E_{\mathbf{w}}[\sum_{i \in N} x_i^p(\mathbf{w}; p)]. \tag{4}$$

Note that the subscript $\mathbf{w}$ of E indicates that the expectation is calculated with respect to $\mathbf{w}$.

C. Totally Incomplete Information

In the totally incomplete information game, each user's true type is private information to him. For the same reason as for the partially incomplete information game, the service provider again charges a uniform unit price p to all the users. Here the solution to the users' game, if exists, is a pure strategy Bayesian equilibrium, which satisfies

$$\begin{aligned} x_i^t(w_i; p) = \arg\max_{x_i} \{ & f(x_i; w_i) \\ & - E_{\mathbf{w_{-i}}} \left[g(x_i, \mathbf{x_{-i}}^t(\mathbf{w_{-i}}; p)) \right] - p x_i \}, \end{aligned} \tag{5}$$

where $\mathbf{w_{-i}} := \{w_j\}_{j \neq i, j \in N}$, and the precise form of g is

$$\begin{aligned} g(x_1^t(w_1; p), \cdots, & x_{i-1}^t(w_{i-1}; p), \\ & x_i, x_{i+1}^t(w_{i+1}; p), \cdots, x_n^t(w_n; p)). \end{aligned}$$

Actually, in this users' game, from the perspective of any user, the other $n-1$ users have independently and identically distributed types with the same distribution; thus, for the Bayesian equilibrium, we should have $x_i^t(w_i; p) = x^t(w_i; p)$ for all $i \in N$. Then for the Stackelberg game, the optimal price for the service provider is

$$p^t = \arg\max_p \; p n E_{w_i}[x^t(w_i; p)], \tag{6}$$

for an arbitrary $i \in N$.

III. Existence of a Unique Equilibrium

Following the approach in [6], we can show for the complete information or partially incomplete information game the existence of a unique Nash equilibrium for the

[1] Here, to avoid the trouble of dealing with non-differentiable functions, we assume that f is twice continuously differentiable. The same assumption holds for other functions such as g, when necessary, throughout the paper.

users' game given the prices. Note that the Nash equilibrium, which solves (1) or (3), actually maximizes

$$\sum_{j\in N} f(x_j;w_j) - g(x_1,\cdots,x_n) - \sum_{j\in N} p_j x_j \tag{7}$$

for each i (with p_j's all equal to p in (7) for partially incomplete information), since the added quantity is not a function of x_i. Therefore, the unique maximizing n-tuple for this common objective function (7), which is strictly concave since f is strictly concave in the first argument and g is convex by assumption, is just the unique Nash equilibrium.

We can also extend the above approach to prove the existence of a unique Bayesian equilibrium. Without loss of generality, we assume that the users' types are discretely distributed, and a continuous distribution is just the limit of a discrete distribution. Suppose that $w_i = w^l$ with probability q_l, for $l \in M := \{1,\cdots,m\}$, where $q_l > 0$ and $\sum_{l=1}^m q_l = 1$. Then, given the prices, the Bayesian equilibrium, which solves (5), is the $(n\times m)$-tuple $\{x_i^{l_i}\}_{i\in N, l_i\in M}$ such that $x_i^{l_i}$ maximizes

$$f(x_i^{l_i};w^{l_i}) - \sum_{\{l_j\}_{j\neq i, l_j\in M}} (\prod_{j\neq i} q_{l_j}) g(x_1^{l_1},\cdots,x_n^{l_n}) - p x_i^{l_i}.$$

By multiplying the above quantity by q_{l_i}, and adding a quantity not relevant to $x_i^{l_i}$, we obtain that $x_i^{l_i}$ equivalently maximizes

$$\begin{aligned}&\sum_{j\in N}\sum_{l_j\in M} q_{l_j}[f(x_j^{l_j};w^{l_j}) - p x_j^{l_j}] \\ &\quad - \sum_{\{l_j\}_{j\in N, l_j\in M}} (\prod_{j\in N} q_{l_j}) g(x_1^{l_1},\cdots,x_n^{l_n}).\end{aligned} \tag{8}$$

Again, this common objective function (8) is strictly concave and admits a unique maximizing $(n\times m)$-tuple, which is the unique Bayesian equilibrium.

IV. Two-User Case

Next, we first compare the partially incomplete information game with the totally incomplete information game. To start with, we consider the simplest case with two symmetric users, and assume that the service provider charges the same fixed unit price p to the two users. Since p is fixed, to save notation, we can simply write the Nash equilibrium as $x_1^p(w_1,w_2)$ and $x_2^p(w_1,w_2)$, and the Bayesian equilibrium as $x_1^t(w_1) = x^t(w_1)$ and $x_2^t(w_2) = x^t(w_2)$.

A. Optimization Problems

For the two users under partially incomplete information, (3) gives the Nash equilibrium. We first consider the following optimization problem:

$$T(y;w) = \arg\max_x \{f(x;w) - g(x,y) - px\}. \tag{9}$$

Note that T is well defined because of the strict concavity of f and the convexity of g. Furthermore, we can easily show that $T(y;w)$ is strictly increasing in w and is decreasing in y. The first argument comes from the assumption that $f(x;w)$ and $\partial f(x;w)/\partial x$ are strictly increasing in w, and the second holds because $g(x,y)$ and $\partial g(x,y)/\partial x$ are increasing in y. This conclusion is consistent with our intuition that a high type user tends to act more aggressively, and an increase in one user's usage has a negative impact on the other user's usage.

Now looking back at (3), we can see that the Nash equilibrium is given by

$$x_1^p(w_1,w_2) = T(x_2^p(w_1,w_2);w_1), \tag{10}$$

$$x_2^p(w_1,w_2) = T(x_1^p(w_1;w_2);w_2), \tag{11}$$

which we know exists and is unique.

On the other hand, for the two users under totally incomplete information, (5) gives the Bayesian equilibrium. Consider the following optimization problem:

$$L(w) = \arg\max_x \{f(x;w) - E_v[g(x,L(v))] - px\}. \tag{12}$$

Then, we have $x^t = L$ and the Bayesian equilibrium is:

$$x_1^t(w_1) = x^t(w_1) = L(w_1),$$
$$x_2^t(w_2) = x^t(w_2) = L(w_2),$$

for which we have already proved the existence and uniqueness. Again, we can easily see that $L(w)$ is strictly increasing in w, since $f(x;w)$ and $\partial f(x;w)/\partial x$ are both strictly increasing in w.

B. Same True Types

First, consider the special case where the two users' true types are the same, i.e., $w_1 = w_2 = w$. Then the two users are completely symmetric in the partially incomplete information game, and for the unique Nash equilibrium, we must have $x_1^p(w,w) = x_2^p(w,w) =: x^p(w)$, which solves

$$x^p(w) = T(x^p(w);w). \tag{13}$$

Obviously, $x^p(w)$ cannot be decreasing in w; otherwise, by (13), $x^p(w)$ would be strictly increasing in w, since $T(y;w)$ is strictly increasing in w and decreasing in y, which leads to a contradiction. Therefore, $x^p(w)$ must be strictly increasing in w. Furthermore, since $T(y;w)$ is decreasing in y, we have the following results: if $x < x^p(w)$, then

$$T(x;w) \geq T(x^p(w);w) = x^p(w) > x;$$

similarly, if $x > x^p(w)$, then $T(x;w) < x$. We summarize this in the following proposition:

Proposition 1: (i) $T(x;w) > x$ if and only if $x < x^p(w)$; (ii) $T(x;w) = x$ if and only if $x = x^p(w)$; and (iii) $T(x;w) < x$ if and only if $x > x^p(w)$.

Now for the totally incomplete information game, the Bayesian equilibrium is $x_1^t(w) = x_2^t(w) = L(w)$, where L solves (12). Note that in the partially incomplete information game, x^p solves (13), where T is given by (9). Thus, in order to compare x^p with L, we want to relate (9) to (12).

Actually, since $L(w)$ is strictly increasing in w and $\partial g(x,y)/\partial x$ is increasing in y, as a result, $\partial g(x,L(w))/\partial x$ is increasing in w. Thus, we can define three sets, $\underline{S}$, S_0 and $\overline{S}$, such that

$$\underline{S} := \{w : \frac{\partial g(x,L(w))}{\partial x} < \frac{\partial}{\partial x} E_v\left[g(x,L(v))\right]\},$$

and S_0 and $\overline{S}$ are similarly defined except that "$<$" in the above relation is changed to "$=$" and "$>$", respectively. Then, we must have $\sup \underline{S} < w_0 < \inf \overline{S}$, for any $w_0 \in S_0$ (provided S_0 is nonempty) [2]. Now, for $w \in \underline{S}$,

$$\frac{\partial g(x, L(w))}{\partial x} < \frac{\partial}{\partial x} E_v \left[g(x, L(v)) \right],$$

and thus by observing (9) and (12), we can easily see that $T(L(w); w) > L(w)$. Similarly, for $w \in S_0$, $T(L(w); w) = L(w)$ and for $w \in \overline{S}$, $T(L(w); w) < L(w)$. Then, immediately from Proposition 1, we conclude the following:

Theorem 1: For $w \in \underline{S}$, $x^p(w) > L(w)$; for $w \in S_0$, $x^p(w) = L(w)$; and for $w \in \overline{S}$, $x^p(w) < L(w)$.

Theorem 1 tells us that when the two users have the same true types, how they behave in the two games depends on whether the types are low or high. Reasonably, if one user's true type is low, then knowing that the other user also has a low type makes him act more aggressively in the partially incomplete information game; on the other hand, if the true types are high, the two users tend to act more aggressively in the totally incomplete information game. Moreover, the results obtained on this special case can help us to compare the two games in a more general context, which follows next.

C. General True Types

Generally, the two users' true types may not necessarily be the same. Then for the partially incomplete information game, x_1 can be computed from (10) and x_2 from (11), alternatively, and the limit is just the unique Nash equilibrium. We can derive several properties of this Nash equilibrium. First, it is obvious that $x_1^p(u, v) = x_2^p(v, u)$, or in other words, the solution is symmetric for the two users. Furthermore, since $T(y; w)$ is strictly increasing in w and is decreasing in y, if we increase w_1, then x_1 strictly increases by (10), and as a result x_2 decreases from (11); then going back to (10), x_1 further increases; and so on. Finally, $x_1^p(w_1, w_2)$ is strictly increasing and $x_2^p(w_1, w_2)$ is decreasing in w_1. On the other hand, if we increase w_2, $x_2^p(w_1, w_2)$ strictly increases and $x_1^p(w_1, w_2)$ decreases. Thus, $x_1^p(w_1, w_2)$ is strictly increasing in w_1 and is decreasing in w_2, while $x_2^p(w_1, w_2)$ is strictly increasing in w_2 and is decreasing in w_1, which is consistent with the above symmetry property.

Recall that for the same true types, $x_1^p(w, w) = x_2^p(w, w) = x^p(w)$, which is strictly increasing in w. Then, we have the following proposition, whose proof is obvious and thus is omitted here:

Proposition 2: For $u < v$, we have $x_1^p(u, v) \le x^p(u) < x^p(v) \le x_1^p(v, u)$, and equivalently, $x_2^p(v, u) \le x^p(u) < x^p(v) \le x_2^p(u, v)$.

Based on Theorem 1 and Proposition 2, we can directly deduce the following theorem, which compares the Nash equilibrium for the partially incomplete information game with the Bayesian equilibrium for the totally incomplete information game:

Theorem 2: For User 1:
(i) if $w_1 \in \underline{S}$ and $w_1 \ge w_2$, then $x_1^p(w_1, w_2) > L(w_1)$;
(ii) if $w_1 \in S_0$ and
 (iia) if $w_1 > w_2$, then $x_1^p(w_1, w_2) \ge L(w_1)$;
 (iib) if $w_1 = w_2$, then $x_1^p(w_1, w_2) = L(w_1)$;
 (iic) if $w_1 < w_2$, then $x_1^p(w_1, w_2) \le L(w_1)$;
(iii) if $w_1 \in \overline{S}$ and $w_1 \le w_2$, then $x_1^p(w_1, w_2) < L(w_1)$.
For User 2, symmetric results hold.

From Theorem 2, we can see that for a user with a comparatively low type, if he knows that the other user's type is not higher, then this information will make him act more aggressively in the partially incomplete information game than in the totally incomplete information game without this information. On the other hand, for a user with a comparatively high type, if the other user's type is not lower, then he will definitely act less aggressively in the partially incomplete information game than in the totally incomplete information game.

From the perspective of the service provider, given a fixed pricing policy, intuitively, if the two users' true types are comparatively low, then they will act more aggressively in the partially incomplete information game, which will lead to a higher profit for the service provider. On the other hand, if the two users' true types are comparatively high, then the service provider may prefer the totally incomplete information game. However, we cannot obtain precise comparison results here, which thus makes it hard to compare the two games under different pricing policies and then to evaluate the service provider's game preference. Analysis of the asymptotic case, however, may provide some insights on this, which will be discussed in the following section.

V. Asymptotic Case

In this section, we consider the asymptotic case for network games, where the user population is very high, as in communication networks. Again, to compare the three classes of games, we first assume that the service provider's pricing policy is fixed.

A. Wardrop Equilibrium

For the asymptotic case, we first discuss the concept of Wardrop equilibrium, which will be applied subsequently to the analysis of the three classes of games. The Wardrop equilibrium originates from [11] to deal with a traffic network where an individual vehicle's impact on the total traffic along the route it takes can be neglected. The Wardrop principle states that "*the journey time on all the routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route*", and this property is referred to as the *delay-equalizing property*. It has been shown in [12] that for a network with a very large number of users, the asymptotic behavior of the Nash equilibrium converges to the Wardrop equilibrium (under appropriate assumptions). Therefore, when analyzing the asymptotic case, the concept of the Wardrop equilibrium can be used instead of the Nash equilibrium to avoid the computational complexity. This idea has been widely and successfully applied to the study of transportation networks since its inception [13], [14].

[2] If $\underline{S}$ is empty, then we let $\sup \underline{S} = -\infty$; if $\overline{S}$ is empty, then $\inf \overline{S} = \infty$.

Telecommunication and communication networks with a large number of users are analogous to transportation networks, and the Wardrop equilibrium has been increasingly used in recent years to study telecommunication networks [15] and communication networks for routing and flow control problems [4], [5]. Particularly in [5], which deals with a combined flow control and routing problem for communication networks, it has been proved that for the given utility functions, the asymptotic Nash equilibrium satisfies the delay-equalizing property as in the Wardrop equilibrium. Henceforth, we apply the Wardrop equilibrium concept to analyze the asymptotic behavior for the flow control problem in this paper.

B. Complete Information

In the complete information game, given $\mathbf{p}$, the Nash equilibrium of the users' game is formulated in (1). Recall that the second term of the objective function, $-g(x_i, \mathbf{x_{-i}}^p(\mathbf{w};p))$, depends on the usage of all users and models one user's loss from the other users' competition, especially, the congestion cost for the case of communication networks. However, as the number of users gets large, i.e., as $n \to \infty$, we can apply the Wardrop equilibrium concept and assume that an individual user's usage is so small compared with the total usage of all users that his contribution to this aggregation term is infinitesimal. In other words, for each individual user, the second term in (1) can be regarded as a constant, and thus we can revise the equilibrium of the users' game for the asymptotic case as:

$$\tilde{x}_i^c(\mathbf{w};\mathbf{p}) = \arg\max_{x_i}\{f(x_i;w_i) - p_i x_i\}, \quad i \in N. \tag{14}$$

In the following, we verify by two examples that the Wardrop equilibrium is actually the limiting solution to the asymptotic case for a special class of utility functions.

Example 1: The first example is taken from [6], where

$$\begin{aligned} f(x_i;w_i) &= w_i \log(1+x_i), \quad i \in N, \\ g(x_1,\cdots,x_n) &= \frac{1}{nc - \sum_{j\in N} x_j}, \end{aligned}$$

for some positive constant c, and each user is charged a uniform price per unit bandwidth p. It has been shown in [6] that as $n \to \infty$, the asymptotic optimal price and users' optimal flows are:

$$\begin{aligned} &p^c(\mathbf{w}) \to \frac{w_{av}}{c+1}, \\ &x_i^c(\mathbf{w};p^c(\mathbf{w})) \to \frac{w_i}{p^c(\mathbf{w})} - 1 = (c+1)\frac{w_i}{w_{av}} - 1, \quad i \in N, \end{aligned}$$

where $w_{av} := \sum_{j\in N} w_j/n$ [3]. On the other hand, if we omit the congestion cost, then the Wardrop equilibrium can be obtained from (14) as

$$\tilde{x}_i^c(\mathbf{w};p) = \frac{w_i}{p} - 1, \quad i \in N.$$

Thus, the optimal price for the service provider is to maximize the total profit $p\sum_{i\in N}\tilde{x}_i^c(\mathbf{w};p)$ subject to the constraint $\sum_{i\in N}\tilde{x}_i^c(\mathbf{w};p) \le nc$ [4], and finally can be obtained as

$$\tilde{p}^c(\mathbf{w}) = \frac{w_{av}}{c+1}.$$

We can see that the solution calculated from the Wardrop equilibrium is consistent with the asymptotic solution calculated from the Nash equilibrium.

Example 2: The second example, taken from [16], extends the uniform pricing scheme in [6] to differentiated pricing such that the service provider may charge a unit price p_i to User i. For the same f and g as defined in [6], it has been shown in [16] that as $n \to \infty$, the asymptotic optimal differentiated prices and users' corresponding optimal flows are:

$$p_i^c(\mathbf{w}) \to \frac{\sqrt{w_i}\,v_{av}^{\frac{1}{2}}}{c+1}, \quad x_i^c(\mathbf{w};\mathbf{p}^c(\mathbf{w})) \to (c+1)\frac{\sqrt{w_i}}{v_{av}^{\frac{1}{2}}} - 1,$$

for $i \in N$, where $v_{av}^{\frac{1}{2}} := \sum_{j\in N}\sqrt{w_j}/n$ [5]. Now without considering the congestion cost, the Wardrop equilibrium from (14) is

$$\tilde{x}_i^c(\mathbf{w};\mathbf{p}) = \frac{w_i}{p_i} - 1, \quad i \in N.$$

Then, the optimal prices for the service provider are to maximize $\sum_{i\in N} p_i\tilde{x}_i^c(\mathbf{w};\mathbf{p})$ subject to $\sum_{i\in N}\tilde{x}_i^c(\mathbf{w};\mathbf{p}) \le nc$. This can be solved by using the Lagrange multiplier method, and finally we obtain

$$\tilde{p}_i^c(\mathbf{w}) = \frac{\sqrt{w_i}\,v_{av}^{\frac{1}{2}}}{c+1}, \quad \tilde{x}_i^c(\mathbf{w};\tilde{\mathbf{p}}^c(\mathbf{w})) = (c+1)\frac{\sqrt{w_i}}{v_{av}^{\frac{1}{2}}} - 1,$$

for $i \in N$. Again, the asymptotic solution from the Nash equilibrium is the same as that from the Wardrop equilibrium.

C. Incomplete Information

Similarly as for the complete information game, we apply the Wardrop equilibrium concept to the partially and totally incomplete information games for the asymptotic case. Then, for the partially incomplete information game, given p, (3) can be revised as

$$\tilde{x}_i^p(\mathbf{w};p) = \arg\max_{x_i}\{f(x_i;w_i) - p x_i\}, \quad i \in N. \tag{15}$$

For the totally incomplete information game, given p, the Bayesian equilibrium is formulated in (5), where the second term of the objective function is the expectation of $-g$. However, for a network with a very large number of users, from the perspective of each individual user, this term can still be regarded as a constant term for the following two reasons. First, compared with the large population, an individual's impact on the total usage of the network is insignificant. Second, as the number of users gets large, the real distribution of all the users' true types should become

[3]Note that we need $w_i > w_{av}/(c+1)$ for User i to be admitted, i.e., have a positive flow, which we assume to be the case here.

[4]Although we do not consider the congestion cost here, it is still required that the total usage cannot exceed the network capacity nc.

[5]Again, we assume that $\sqrt{w_i} > v_{av}^{\frac{1}{2}}/(c+1)$ for $i \in N$.

more and more concentrated on some certain pattern which is consistent with q, and thus an individual user can reasonably neglect the variance of distribution of the other users' types. Therefore, finally, (5) can be revised as

$$\tilde{x}_i^t(w_i;p) = \arg\max_{x_i}\{f(x_i;w_i) - px_i\}, \quad i \in N. \qquad (16)$$

Comparing (15) with (16), we can see that given a fixed price p, the Nash equilibrium for partially incomplete information and the Bayesian equilibrium for totally incomplete information actually become the same in the limiting case. As a result, as $n \to \infty$, the optimization problems to solve for the asymptotic optimal prices for the service provider, given by (4) and (6) for partially and totally incomplete information, respectively, actually become the same problem. In other words, the distinction between partially incomplete information and totally incomplete information vanishes as the user population grows large, and thus we can refer to both games just as the incomplete information game.

Intuitively, in a network with a very large number of users, the aggregate usage and thus the network performance remain comparably stable. Therefore, each individual user just determines his optimal usage based on his own type and the price, and as a result the knowledge of the other users' true types in the partially incomplete information game is of no value to him.

D. Service Provider's Game Preference

Next, we compare the complete information game with the incomplete information game for the asymptotic case from the perspective of the service provider. These two games are different in two aspects. First, by observing (14) and (15), we can see that the service provider can differentiate prices based on the users' types in the complete information game, while for incomplete information, he has to charge a uniform price to all users due to lack of information on each user's true type. Second, by observing (2) and (4), in the complete information game, the service provider determines his optimal prices such that the profit is maximized, while in the incomplete information game, his objective function is the expected profit. However, as the number of users gets large, as mentioned previously, the real distribution of all the users' true types should become more and more concentrated on some certain pattern, and thus the service provider can neglect the variance of this distribution. Therefore, the effect of the second distinction of the two games disappears for the asymptotic case. Then, it becomes obvious that the service provider prefers the complete information game, where he may get a higher profit by price differentiation [6].

VI. Conclusion and Extensions

In this study, we have focused on understanding how a monopolistic service provider and multiple users perform in network games with various types of public and private information. Specifically, we have shown that for a network with a large number of users, the distinction between the partially incomplete information game and the totally incomplete information game is inconsequential, while the service provider prefers the complete information game to the incomplete information game for possibly higher profit. This result provides useful insight for communication network applications.

This work can be extended in several directions. First, for the asymptotic case, other than the two examples which show for the complete information game that the limiting Nash equilibrium does converge to the Wardrop equilibrium for a special class of utility functions, applying the Wardrop equilibrium to solve for the asymptotic solution is rather intuitive. In the future, we wish to obtain a general proof of this convergence for the three classes of games. Also, in this paper, we have assumed that the service provider has a linear pricing policy, i.e., each user is charged a fixed unit price. This can be relaxed to include nonlinear pricing policies as well [17]. Finally, network games with multiple competitive service providers need to be studied in the future.

[6] For a comparison of the uniform pricing and differentiated pricing schemes, see [16].

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