Hierarchical Network Games with Various Types of Public and Private Information

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Motivation

- Hierarchical network game
  - Stackelberg (leader-follower) game
    single service provider — leader
    multiple users — followers
  - Noncooperative game among the users
- A user's type:
  - public information (shared with all the players)
    → Complete information
  - private information (only to that user)
    → Totally incomplete information
  - common information only among the users
    → Partially incomplete information
- Goal: for the three classes of games,
  study equilibria and compare performances of the players

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Previous Work

- Başar and Srikant INFOCOM'02, JOTA'02
  proposed the hierarchical Stackelberg game model
  studied uniform pricing
- Shen and Başar ACC'04
  studied differentiated pricing
- Shen and Başar CDC'04
  extended complete information to incomplete information
- This work: extends to
  general utility functions
  general distributions for user characteristics

Outline

Problem Formulation

- Set of users: $N = \{1, \cdots, n\}$; flow of User i: x_i, $i \in N$
- Price per unit capacity charged to User i: p_i
- User i's net utility: $f(x_i; w_i) - g(x_1, \cdots, x_n) - p_i x_i$
- Service provider's revenue: $\sum_{j \in N} p_j x_j$
- Users' types (w_i's):
  independently and identically distributed
  common distribution publicly known to all the players
- Notations: $\mathbf{w} := \{w_j\}_{j \in N}$, $\mathbf{x} := \{x_j\}_{j \in N}$, $\mathbf{p} := \{p_j\}_{j \in N}$,
  $\mathbf{w}_{-i} := \{w_j\}_{j \neq i, j \in N}$, $\mathbf{x}_{-i} := \{x_j\}_{j \neq i, j \in N}$.

Complete Information

Nash equilibrium

Given $\mathbf{p}$:

$$x_i^c(\mathbf{w};\mathbf{p}) = \arg\max_{x_i}\{f(x_i;w_i) - g(x_i,\mathbf{x_{-i}}^c(\mathbf{w};\mathbf{p})) - p_i x_i\},\ \ i \in N.$$

Stackelberg game solution

$$\mathbf{p}^c(\mathbf{w}) = \arg\max_{\mathbf{p}} \sum_{j\in N} p_j x_j^c(\mathbf{w};\mathbf{p}),\quad \mathbf{x}^c(\mathbf{w};\mathbf{p}^c(\mathbf{w})).$$

Partially Incomplete Information

Nash equilibrium

Given $p_i = p$, $i \in N$:

$$x_i^p(\mathbf{w}; p) = \arg\max_{x_i} \{f(x_i; w_i) - g(x_i, \mathbf{x_{-i}}^p(\mathbf{w}; p)) - px_i\}.$$

Stackelberg game solution

$$p^p = \arg\max_p pE_{\mathbf{w}}[\sum_{j \in N} x_j^p(\mathbf{w}; p)], \quad \mathbf{x}^p(\mathbf{w}; p^p).$$

Totally Incomplete Information

Bayesian equilibrium

Given $p_i = p$, $i \in N$:

$$x_i^t(w_i; p) = \arg\max_{x_i} \{ f(x_i; w_i) - E_{\mathbf{w}_{-i}}[g(x_i, \mathbf{x}_{-i}{}^t(\mathbf{w}_{-i}; p))] - p x_i \}.$$

Stackelberg game solution

$$p^t = \arg\max_p pnE_{w_i}[x_i^t(w_i; p)], \quad \{x_i^t(w_i; p^t)\}_{i \in N}.$$

Existence of a Unique Nash Equilibrium

Theorem

For the complete information or partially incomplete information game, given the prices, there exists a unique Nash equilibrium.

Proof

1. For User i, $i \in N$, the following two are equivalent:

$$\max_{x_i} f(x_i; w_i) - g(x_1, \cdots, x_n) - p_i x_i,$$

$$\max_{x_i} \sum_{j \in N} f(x_j; w_j) - g(x_1, \cdots, x_n) - \sum_{j \in N} p_j x_j.$$

2. The common objective function is strictly concave, and the unique maximizing n-tuple is the unique Nash equilibrium.

Existence of a Unique Bayesian Equilibrium

Theorem

For the totally incomplete information game, given the prices, there exists a unique Bayesian equilibrium.

Proof

Assume $w_i = w^l$ w.p. q_l, $l \in M$. For User i with the type w^{l_i}, $i \in N$, $l_i \in M$, the following two are equivalent:

$$\max_{x_i^{l_i}} f(x_i^{l_i}; w^{l_i}) - \sum_{\{l_j\}_{j \neq i, l_j \in M}} (\prod_{j \neq i} q_{l_j}) g(x_1^{l_1}, \cdots, x_n^{l_n}) - p x_i^{l_i},$$

$$\max_{x_i^{l_i}} \sum_{j \in N} \sum_{l_j \in M} q_{l_j} [f(x_j^{l_j}; w^{l_j}) - p x_j^{l_j}] - \sum_{\{l_j\}_{j \in N, l_j \in M}} (\prod_{j \in N} q_{l_j}) g(x_1^{l_1}, \cdots, x_n^{l_n}).$$

Two Users with the Same True Types

Three sets of types: $\underline{S}$, S_0 and $\overline{S}$ ($\sup \underline{S} < w_0 \in S_0 < \inf \overline{S}$)

$$\begin{aligned} w \in \overline{S} \quad &\Rightarrow \quad x_i^p(\mathbf{w}; p) < x_i^t(w; p) \\ w \in S_0 \quad &\Rightarrow \quad x_i^p(\mathbf{w}; p) = x_i^t(w; p) \\ w \in \underline{S} \quad &\Rightarrow \quad x_i^p(\mathbf{w}; p) > x_i^t(w; p) \end{aligned}$$

Partially vs Totally incomplete information

high types $\Rightarrow$
more aggressive users under totally incomplete information
low types $\Rightarrow$
more aggressive users under partially incomplete information

Two Users with General True Types

$$w_1 \in \overline{S} \quad w_1 \leq w_2 \quad \Rightarrow \quad x_1^p(\mathbf{w}; p) < x_1^t(w_1; p)$$

$$w_1 \in S_0 \quad \begin{array}{lcl} w_1 < w_2 & \Rightarrow & x_1^p(\mathbf{w}; p) \leq x_1^t(w_1; p) \\ w_1 = w_2 & \Rightarrow & x_1^p(\mathbf{w}; p) = x_1^t(w_1; p) \\ w_1 > w_2 & \Rightarrow & x_1^p(\mathbf{w}; p) \geq x_1^t(w_1; p) \end{array}$$

$$w_1 \in \underline{S} \quad w_1 \geq w_2 \quad \Rightarrow \quad x_1^p(\mathbf{w}; p) > x_1^t(w_1; p)$$

Partially vs Totally incomplete information

high type User 1, higher type User 2 $\Rightarrow$
more aggressive User 1 under totally incomplete information
low type User 1, lower type User 2 $\Rightarrow$
more aggressive User 1 under partially incomplete information

Service Provider's Game Preference

Conclusions

- Intuitively, high type users tend to act more aggressively under totally incomplete information, which leads to a higher profit for the service provider.
- On the other hand, when users have comparatively low types, the service provider may prefer the partially incomplete information game for a higher profit.
- For a small number of users, precise comparison results are hard to obtain, even with a fixed pricing policy.

Asymptotic Case

- Communication networks: a large number of users
- Wardrop equilibrium
  - transportation networks
    neglect an individual vehicle's impact on the total traffic
  - networks with a large number of users
    Nash equilibrium → Wardrop equilibrium (Haurie and Marcotte '85)
  - communication networks
    routing and flow control (Altman, Başar and Srikant '02, Acemoglu and Ozdaglar '06)

Wardrop Equilibrium

- As $n \to \infty$, for User i, regard $g(x_1, \cdots, x_n)$ as a constant.
- Given the prices, revise the users' game as follows: complete information: $\max_{x_i} f(x_i; w_i) - p_i x_i$, $i \in N$; incomplete information: $\max_{x_i} f(x_i; w_i) - p x_i$, $i \in N$.
- For a special class of utility functions (as in Başar and Srikant INFOCOM'02):

$$f(x_i; w_i) = w_i \log(1 + x_i), \quad i \in N,$$

$$g(x_1, \cdots, x_n) = \frac{1}{nc - \sum_{j \in N} x_j},$$

the asymptotic behavior of the Nash / Bayesian equilibrium converges to the Wardrop equilibrium.

Service Provider's Game Preference

For the special class of utility functions:

Conclusions

- The service provider makes the highest profit under complete information, since he can charge differentiated prices according to the users' true types.
- The service provider makes a higher expected profit under partially incomplete information; under totally incomplete information, he charges more conservatively (with a higher optimal price) to guarantee that the total capacity is not exceeded.

Extensions

- Asymptotic case: convergence to Wardrop equilibrium for general utility functions
- Nonlinear pricing (with quantity discount)
- Multiple service providers