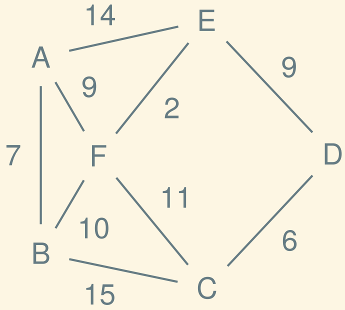
Single-Source Shortest Paths

CS 214, Fall 2019

The problem

Find the shortest path from A to D:



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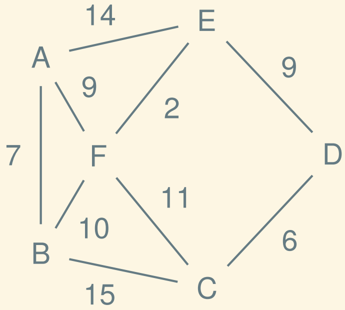

Typically generalized as single-source shortest paths (SSSP): find the shortest path to everywhere from A.

Relaxation idea

Keep a table with two values for each node:

- the best known distance to it from the start, and
- the predecessor node along that best path

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For example, suppose that

- the best known distance to node C is 15,
- the best known distance to node D is 4, and
- there’s an edge of weight 5 from D to C.

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- the predecessor node along that best path

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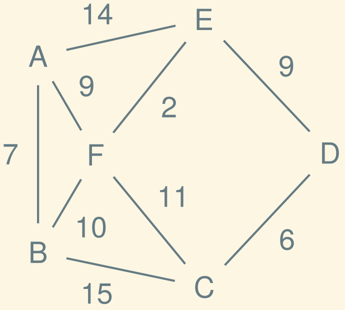
For example, suppose that

- the best known distance to node C is 15,
- the best known distance to node D is 4, and
- there’s an edge of weight 5 from D to C.

Then we update the best known distance to C to be 9, via D.

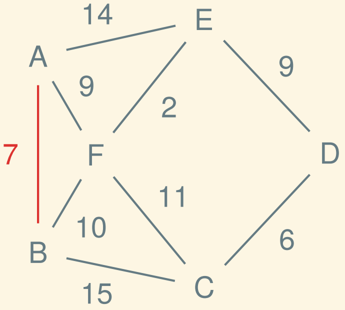
Relaxation demonstration

v	dist[v]	pred[v]
A	0	
B	∞	
C	∞	
D	∞	
E	∞	
F	∞	



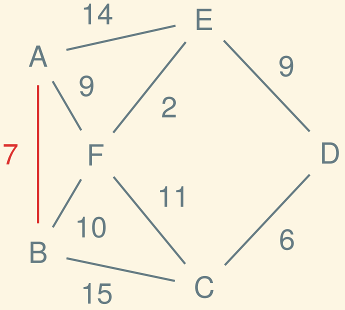
Relaxation demonstration

v	dist[v]	pred[v]
A	0	
B	∞	
C	∞	
D	∞	
E	∞	
F	∞	



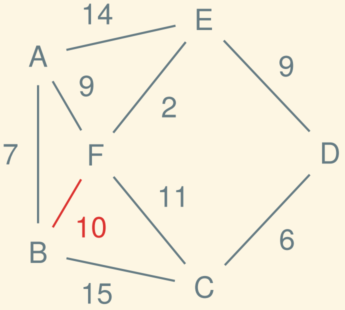
Relaxation demonstration

v	dist[v]	pred[v]
A	0	
B	7	A
C	∞	
D	∞	
E	∞	
F	∞	



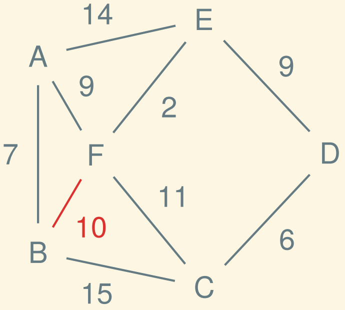
Relaxation demonstration

v	dist[v]	pred[v]
A	0	
B	7	A
C	∞	
D	∞	
E	∞	
F	∞	



Relaxation demonstration

v	dist[v]	pred[v]
A	0	
B	7	A
C	∞	
D	∞	
E	∞	
F	17	B



Relaxation demonstration

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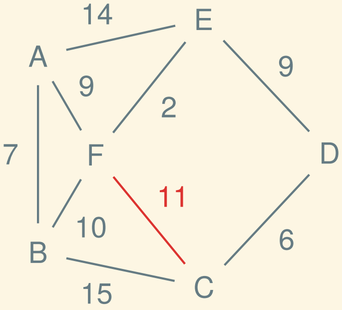
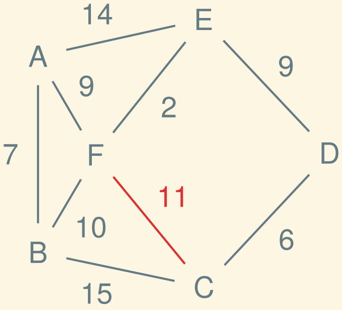
A	0	
B	7	A
C	∞	
D	∞	
E	∞	
F	17	B



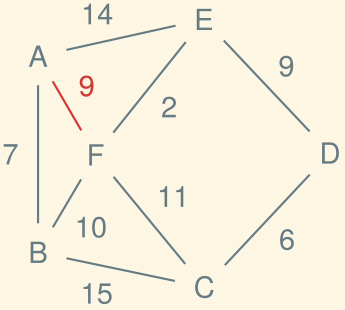
Relaxation demonstration

v	dist[v]	pred[v]
A	0	
B	7	A
C	28	F
D	∞	
E	∞	
F	17	B



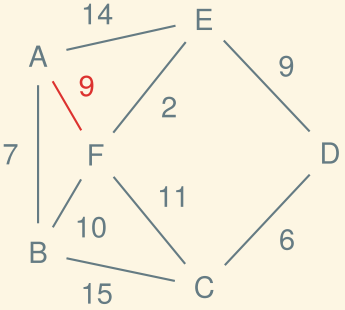
Relaxation demonstration

v	dist[v]	pred[v]
A	0	
B	7	A
C	28	F
D	∞	
E	∞	
F	17	B



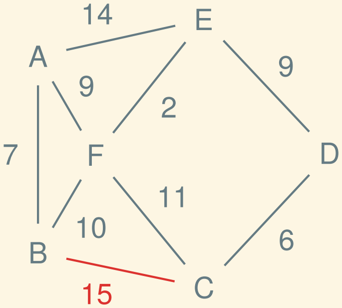
Relaxation demonstration

v	dist[v]	pred[v]
A	0	
B	7	A
C	28	F
D	∞	
E	∞	
F	9	A



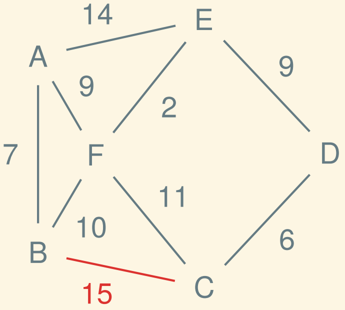
Relaxation demonstration

v	dist[v]	pred[v]
A	0	
B	7	A
C	28	F
D	∞	
E	∞	
F	9	A

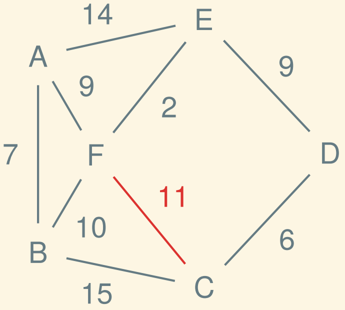


Relaxation demonstration

v	dist[v]	pred[v]
A	0	
B	7	A
C	22	B
D	∞	
E	∞	
F	9	A



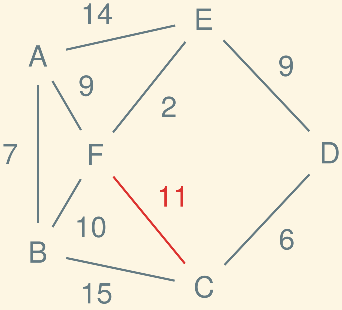
Relaxation demonstration



v	dist[v]	pred[v]
A	0	
B	7	A
C	22	B
D	∞	
E	∞	
F	9	A

Relaxation demonstration

v	dist[v]	pred[v]
A	0	
B	7	A
C	20	F
D	∞	
E	∞	
F	9	A



Bellman–Ford algorithm summary

Solves:	SSSP for graphs with no negative cycles
Main idea:	Relax every edge $|V| - 1$ times
Time complexity:	$\mathcal{O}(VE)$

The Bellman–Ford algorithm

Input: A graph *graph* and a starting vertex *start*
Output: Tables of vertex distances *dist* and predecessors *pred*

```
for every vertex v in graph do
    dist[v] ← ∞;
    pred[v] ← −1
end
dist[start] ← 0;

for |Vertices(graph)| − 1 iterations do
    for every edge (v,u) with weight w in graph do
        if dist[v] + w < dist[u] then
            dist[u] ← dist[v] + w;
            pred[u] ← v
        end
    end
end
```

continued...

Bellman–Ford, continued

At this point we have the answer provided there are no negative-weight cycles. We do one more pass to ensure this is the case:

for every edge (v,u) with weight w **in** *graph* **do**
    if $dist[v] + w < dist[u]$ **then**
        graph contains a negative cycle!
    end
end

Dijkstra's algorithm summary

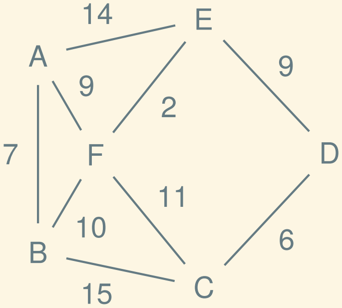
Solves:	SSSP for graphs with no negative *edges*
Main idea:	Relax the edges in a clever order
Time complexity:	depends

Dijkstra's algorithm summary

Solves:	SSSP for graphs with no negative *edges*
Main idea:	Relax the edges in a clever order
Time complexity:	depends

What's the clever order? Relax the edges coming out of the nearest vertex, then repeat

Dijkstra's algorithm demonstration



v	dist[v]	pred[v]
A	0	
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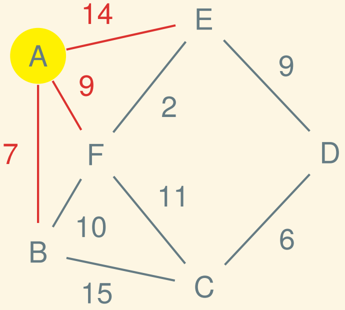
C	∞	
D	∞	
E	∞	
F	∞	

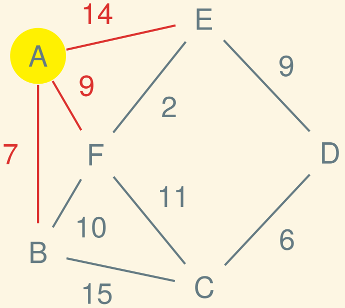
Dijkstra's algorithm demonstration



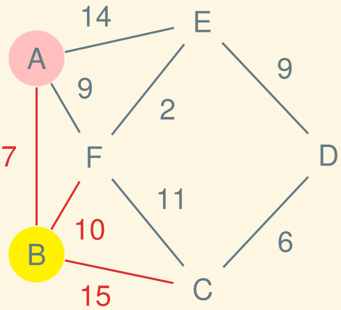
v	dist[v]	pred[v]
A	0	
B	∞	
C	∞	
D	∞	
E	∞	
F	∞	

Dijkstra's algorithm demonstration

v	dist[v]	pred[v]
A	0	
B	7	A
C	∞	
D	∞	
E	14	A
F	9	A



Dijkstra's algorithm demonstration



v	dist[v]	pred[v]
A	0	
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A	0	
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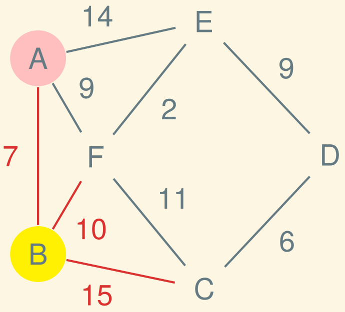

Dijkstra's algorithm demonstration

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C	22	B
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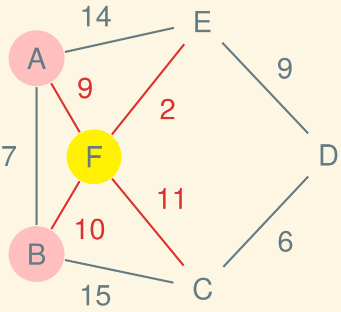
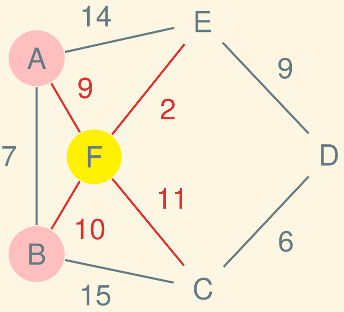
F	9	A



Dijkstra's algorithm demonstration

v	dist[v]	pred[v]
A	0	
B	7	A
C	20	F
D	∞	
E	11	F
F	9	A



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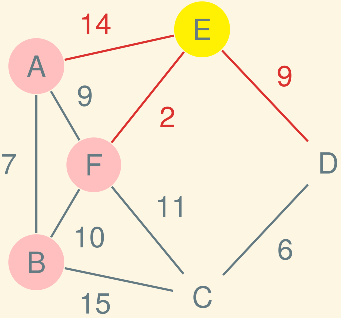

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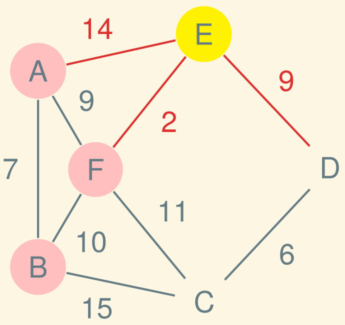

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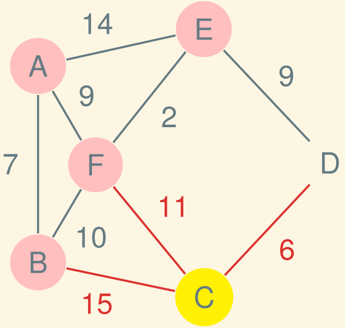

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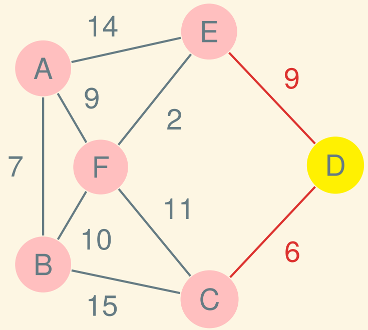

Dijkstra's algorithm demonstration

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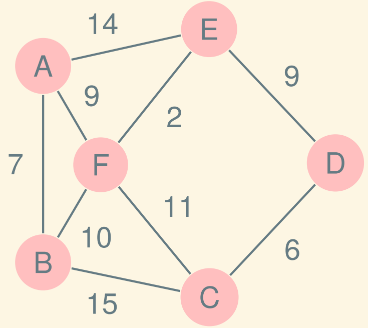
C	20	F
D	20	E
E	11	F
F	9	A



Dijkstra's algorithm (original)

Input: A graph *graph* and a starting vertex *start*
Output: Tables of vertex distances *dist* and predecessors *pred*

```
for every vertex v in graph do
    dist[v] ← ∞; pred[v] ← −1;
end
dist[start] ← 0;

todo ← the set of vertices in graph;
while todo is not empty do
    v ← remove the element of todo with minimal dist[v];
    for every outgoing edge (v,u) with weight w do
        if dist[v] + w < dist[u] then
            dist[u] ← dist[v] + w;
            pred[u] ← v
        end
    end
end
```

Priority Queue ADT

Looks like: ⟨ 2:g 5:i 5:b 17:c 89:g ⟨ (note sorting)

```
struct key_value:
    let key
    let value

interface PRIORITY_QUEUE:
    def is_empty(self) -> bool?
    def insert(self, key: num?, value: AnyC) -> VoidC
    def peek_min(self) -> key_value?
    def remove_min(self) -> key_value?
```

Behavior:

- Keeps key-value pairs sorted by key, so that
- `remove_min` can find and remove the pair with the smallest key

Dijkstra's algorithm with priority queue (1/2)

Input: A graph *graph* and a starting vertex *start*
Output: Tables of vertex distances *dist* and predecessors *pred*

for every vertex *v* in *graph* **do**
    dist$[v] \leftarrow \infty$; *pred*$[v] \leftarrow -1$;
end
dist$[start] \leftarrow 0$;
done $\leftarrow$ empty vertex set;
todo $\leftarrow$ empty priority queue;
`Insert(`*todo, 0, start*`)`;

Dijkstra's algorithm with priority queue (2/2)

while *todo* is not empty **do**
    $(\_, v) \leftarrow$ `RemoveMin`(*todo*);
    if $v \notin done$ **then**
        $done \leftarrow done \cup \{v\}$;
        for every outgoing edge (v,u) with weight w **do**
            if $dist[v] + w < dist[u]$ **then**
                $dist[u] \leftarrow dist[v] + w$;
                $pred[u] \leftarrow v$;
                `Insert`(*todo*, $dist[u]$, u)
            end
        end
    end
end

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- Relax every edge once, for $\mathcal{O}(E)$
- For every edge, we (might) do an `insert`, which takes how long?

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- For every edge, we (might) do an `remove_min`, which takes how long? Call it T_{rm}.

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- For every edge, we (might) do an `insert`, which takes how long? Call it T_{in}.
- For every edge, we (might) do an `remove_min`, which takes how long? Call it T_{rm}.

- Then Dijkstra's algorithm is $\mathcal{O}(E(T_{in} + T_{rm}))$.

Next: making `remove_min` and `insert` fast