Section 5.2

5.8. Two dice are tossed.

a. Let X_1 be a random variable that is numerically equal to the total number of spots on the up faces of the dice. Construct a table that defines this random variable.

b. Let X_2 be a random variable that has the value of 1 if the sum of the number of spots up on both dice is even and the value zero if it is odd. Repeat part (a) for this case.

5.9. Three fair coins are tossed simultaneously such that they don't interact. Define a random variable $X = 1$ if an even number of heads is up and $X = 0$ otherwise. Plot the cumulative distribution function and the probability density function corresponding to this random variable.

5.10. A certain *continuous* random variable has the cumulative distribution function

$$F_X(x) = \begin{cases} 0, & x < 0 \\ Ax^4, & 0 \le x \le 12 \\ B, & x > 12 \end{cases}$$

a. Find the proper values for A and B.

b. Obtain and plot the pdf $f_X(x)$.

c. Compute $P(X > 5)$.

d. Compute $P(4 \le X < 6)$.

5.11. The following functions can be pdfs if constants are chosen properly. Find the proper conditions on the constants [$A, B, C, D, \alpha, \beta, \gamma$, and τ are positive constants, and $u(x)$ is the unit step function.]

a. $f(x) = Ae^{-\alpha x}u(x)$, where $u(x)$ is the unit step.

b. $f(x) = Be^{\beta x}u(-x)$.

c. $f(x) = Ce^{-\gamma x}u(x-1)$.

d. $f(x) = D[u(x) - u(x-\tau)]$.

5.12. Test X and Y for independence if

a. $f_{XY}(x,y) = Ae^{-|x|-2|y|}$.

b. $f_{XY}(x,y) = C(1-x-y), 0 \le x \le 1-y$ and $0 \le y \le 1$.

Prove your answers.

5.13. The joint pdf of two random variables is

$$f_{XY}(x,y) = \begin{cases} C(1+xy), & 0 \le x \le 4, 0 \le y \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the following:

a. The constant C.

b. $f_{XY}(1, 1.5)$.

c. $f_{XY}(x, 3)$.

d. $f_{X|Y}(x|1)$.

5.14. The joint pdf of the random variables X and Y is

$$f_{XY}(x,y) = Axye^{-(x+y)}, \quad x \ge 0 \text{ and } y \ge 0$$

a. Find the constant A.

b. Find the marginal pdfs of X and Y, $f_X(x)$ and $f_Y(y)$.

c. Are X and Y statistically independent? Justify your answer.

5.15.

a. For what value of $\alpha > 0$ is the function

$$f(x) = \alpha x^{-2}u(x-\alpha)$$

a probability density function? Use a sketch to illustrate your reasoning and recall that a pdf has to integrate to 1. [$u(x)$ is the unit step function.]

b. Find the corresponding cumulative distribution function.

c. Compute $P(X \ge 10)$.

5.16. Given the Gaussian random variable with the pdf

$$f_X(x) = \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi}\sigma}$$

where $\sigma > 0$ is the standard deviation. If $Y = X^2$ find the pdf of Y. *Hint*: Note that $Y = X^2$ is symmetrical about $X = 0$ and that it is impossible for Y to be less than zero.

5.17. A nonlinear system has input X and output Y. The pdf for the input is Gaussian as given in Problem 5.16. Determine the pdf of the output, assuming that the nonlinear system has the following input–output relationship:

a. $Y = \begin{cases} aX, & X \ge 0 \\ 0, & X < 0 \end{cases}$

Hint: When $X < 0$, what is Y? How is this manifested in the pdf for Y?

b. $Y = |X|$.

c. $Y = X - X^3/3$.

Section 5.3

5.18. Let $f_X(x) = A\exp(-bx)u(x-2)$ for all x where A and b are positive constants.

a. Find the relationship between A and b such that this function is a pdf.

b. Calculate $E(X)$ for this random variable.

c. Calculate $E(X^2)$ for this random variable.

d. What is the variance of this random variable?

5.19.

a. Consider a random variable uniformly distributed between 0 and 2. Show that $E(X^2) > E^2(X)$.

b. Consider a random variable uniformly distributed between 0 and 4. Show that $E(X^2) > E^2(X)$.

c. Can you show in general that for any random variable it is true that $E(X^2) > E^2(X)$ unless the random variable is zero almost always? (*Hint:* Expand $E[(X - E[X])^2] \geq 0$, and note that it is 0 only if $X = 0$ with probability 1.)

5.20. Verify the entries in Table 5.5 for the mean and variance of the following probability distributions:

a. Rayleigh

b. One-sided exponential

c. Hyperbolic

d. Poisson

e. Geometric

5.21. A random variable X has the pdf

$$f_X(x) = Ae^{-bx}[u(x) - u(x - B)]$$

where $u(x)$ is the unit step function and A, B, *and* b are positive constants.

a. Find the proper relationship between the constants $A, b,$ and B. Express b in terms of A and B.

b. Determine and plot the cdf.

c. Compute $E[X]$.

d. Determine $E[X^2]$.

e. What is the variance of X?

5.22. If

$$f_X(x) = \left(2\pi\sigma^2\right)^{-1/2}\exp\left(-\frac{x^2}{2\sigma^2}\right)$$

show that

a. $E[X^{2n}] = 1 \times 3 \times 5 \times \ldots (2n - 1)\sigma^{2n}$, for $n = 1, 2, \ldots$

b. $E[X^{2n-1}] = 0$ for $n = 1, 2, \ldots$

5.23. The random variable has pdf

$$f_X(x) = \frac{1}{2}\delta(x - 5) + \frac{1}{8}[u(x - 4) - u(x - 8)]$$

where $u(x)$ is the unit step. Determine the mean and the variance of the random variable thus defined.

5.24. Two random variables X and Y have means and variances given below:

$$m_x = 1,\ \sigma_x^2 = 4,\ m_y = 3,\ \sigma_y^2 = 7$$

A new random variable Z is defined as

$$Z = 3X - 4Y$$

Determine the mean and variance of Z for each of the following cases of correlation between the random variables X and Y:

a. $\rho_{XY} = 0$.

b. $\rho_{XY} = 0.2$.

c. $\rho_{XY} = 0.7$.

d. $\rho_{XY} = 1.0$.

5.25. Two Gaussian random variables X and Y, with zero means and variances σ^2, between which there is a correlation coefficient ρ, have a joint probability density function given by

$$f(x,y) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}}\exp\left(-\frac{x^2 - 2\rho xy + y^2}{2\sigma^2(1-\rho^2)}\right)$$

The marginal pdf of Y can be shown to be

$$f_Y(y) = \frac{\exp(-y^2/2\sigma^2)}{\sqrt{2\pi\sigma^2}}$$

Find the conditional pdf $f_{X|Y}(x \mid y)$. Simplify.

5.26. Using the definition of a conditional pdf given by (5.62) and the expressions for the marginal and joint Gaussian pdfs, show that for two jointly Gaussian random variables X and Y, the conditional density function of X given Y has the form of a Gaussian density with conditional mean and the conditional variance given by

$$E[X|Y] = m_x + \frac{\rho\sigma_x}{\sigma_y}(Y - m_y)$$

and

$$\text{var}(X|Y) = \sigma_x^2(1 - \rho^2)$$

respectively.

5.27. The random variable X has a probability density function uniform in the range $0 \leq x \leq 2$ and zero elsewhere. The independent variable Y has a density uniform in the range $1 \leq y \leq 5$ and zero elsewhere. Find and plot the density of $Z = X + Y$.

5.28. A random variable X is defined by

$$f_X(x) = 4e^{-8|x|}$$

The random variable Y is related to X by $Y = 4 + 5X$.

a. Determine $E[X], E[X^2]$, and σ_x^2.

b. Determine $f_Y(y)$.

c. Determine $E[Y], E[Y^2]$, and σ_y^2. (Hint: The result of part (b) is not necessary to do this part, although it may be used)

5.29. A random variable X has the probability density function

$$f_X(x) = \begin{cases} ae^{-ax}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

where a is an arbitrary positive constant.

a. Determine the characteristic function $M_x(jv)$.

b. Use the characteristic function to determine $E[X]$ and $E[X^2]$.

c. Check your results by computing

$$\int_{-\infty}^{\infty} x^n f_X(x)\, dx$$

for $n = 1$ and 2.

d. Compute σ_x^2.

Section 5.4

5.30. Compare the binomial, Laplace, and Poisson distributions for

a. $n = 3$ and $p = \frac{1}{5}$.

b. $n = 3$ and $p = \frac{1}{10}$.

c. $n = 10$ and $p = \frac{1}{5}$.

d. $n = 10$ and $p = \frac{1}{10}$.

5.31. An honest coin is flipped 10 times.

a. Determine the probability of the occurrence of either five or six heads.

b. Determine the probability of the first head occurring at toss number 5.

c. Repeat parts (a) and (b) for flipping 100 times and the probability of the occurrence of 50 to 60 heads inclusive and the probability of the first head occurring at toss number 50.

5.32. Passwords in a computer installation take the form $X_1X_2X_3X_4$, where each character X_i is one of the 26 letters of the alphabet. Determine the maximum possible number of different passwords available for assignment for each of the two following conditions:

a. A given letter of the alphabet can be used only once in a password.

b. Letters can be repeated if desired, so that each X_i is completely arbitrary.

c. If selection of letters for a given password is completely random, what is the probability that your competitor could access, on a single try, your computer in part (a)? and part (b)?

5.33. Assume that 20 honest coins are tossed.

a. By applying the binomial distribution, find the probability that there will be fewer than three heads.

b. Do the same computation using the Laplace approximation.

c. Compare the results of parts (a) and (b) by computing the percent error of the Laplace approximation.

5.34. A digital data transmission system has an error probability of 10^{-5} per digit.

a. Find the probability of exactly one error in 10^5 digits.

b. Find the probability of exactly two errors errors in 10^5 digits.

c. Find the probability of more than five errors in 10^5 digits.

5.35. Assume that two random variables X and Y are jointly Gaussian with $m_x = m_y = 1$, $\sigma_x^2 = \sigma_y^2 = 4$, and correlation coeficient $\rho = 0.5$.

a. Making use of (5.194), write down an expression for the margininal pdfs of X and of Y.

b. Write down an expression for the conditional pdf $f_{X|Y}(x|y)$ by using the result of (a) and an expression for $f_{XY}(x, y)$ written down from (5.189). Deduce that $f_{Y|X}(y|x)$ has the same form with y replacing x.

c. Put $f_{X|Y}(x|y)$ into the form of a marginal Gaussian pdf. What is its mean and variance? (The mean will be a function of y.)

5.36. Consider the Cauchy density function

$$f_X(x) = \frac{K}{1+x^2}, \quad -\infty \leq x \leq \infty$$

a. Find K.

b. Show that var$[X]$ is not finite.

c. Show that the characteristic function of a Cauchy random variable is $M_x(jv) = \pi K e^{-|v|}$.

d. Now consider $Z = X_1 + \ldots + X_N$ where the X_i's are independent Cauchy random variables. Thus their characteristic function is

$$M_Z(jv) = (\pi K)^N \exp(-N|v|)$$