HW9 Solutions

Problem 6.14

a. $E\left[X^2(t)\right] = R(0) = 12$ W; $E^2[X(t)] = \lim_{\tau\to\infty} R(\tau) = 9$ W;

$\sigma_X^2 = E\left[X^2(t)\right] - E^2[X(t)] = 12 - 9 = 3$ W.

b. DC power $= E^2[X(t)] = 9$ W.

c. Total power $= E\left[X^2(t)\right] = 12$ W.

d. $S(f) = 9\delta(f) + 15\,\text{sinc}^2(5f)$.

Problem 6.15

a. The autocorrelation function is

$$\begin{aligned} R_X(\tau) &= E[Y(t)Y(t+\tau)] \\ &= E\{[X(t) + X(t-T)][X(t+\tau) + X(t+\tau-T)]\} \\ &= E[X(t)X(t+\tau)] + E[X(t)X(t+\tau+\tau)] \\ &\quad + E[X(t-T)X(t+\tau)] + E[X(t-T)X(t+\tau-T)] \\ &= 2R_X(\tau) + R_X(\tau-T) + R_X(\tau+T) \end{aligned}$$

b. Application of the time delay theorem of Fourier transforms gives

$$\begin{aligned} S_Y(f) &= 2S_X(f) + S_X(f)[\exp(-j2\pi fT) + \exp(j2\pi fT)] \\ &= 2S_X(f) + 2S_X(f)\cos(2\pi fT) \\ &= 4S_X(f)\cos^2(\pi fT) \end{aligned}$$

c. Use the transform pair

$$R_X(\tau) = 5\Lambda(\tau) \longleftrightarrow S_X(f) = 5\,\text{sinc}^2(f)$$

and the result of (b) to get

$$\cancel{S_Y(f) = 24\,\text{sinc}^2(f)\cos^2(\pi f/4)}$$

$$S_Y(f) = 20\,\text{sinc}^2(f)\cos^2\left(\frac{\pi f}{2}\right)$$

Problem 6.23

a. $E[Y(t)] = 0$ because the mean of the input is zero.

b. The frequency response function of the filter is

$$H(f) = \frac{1}{10 + j2\pi f}$$

The output power spectrum is

$$\begin{aligned} S_Y(f) &= S_x(f) |H(f)|^2 \\ &= 1 \times \frac{1}{100 + (2\pi f)^2} \\ &= \frac{0.01}{1 + (2\pi f/10)^2} = 0.05 \times \frac{2/10}{1 + (2\pi f/10)^2} \end{aligned}$$

which is obtained applying (6.89).

c. Use the transform pair $\exp(-|\tau|/\tau_0) \longleftrightarrow \frac{2\tau_0}{1+(2\pi f \tau_0)^2}$ to find the power spectrum as

$$R_Y(\tau) = 0.05 e^{-10|\tau|}$$

d. Since the input is Gaussian, so is the output. Also, $E[Y] = 0$ and $\text{var}[Y] = R_Y(0) = 0.05$, so

$$f_Y(y) = \frac{\exp(-y^2/2\sigma_Y^2)}{\sqrt{2\pi\sigma_Y^2}}$$

where $\sigma_Y^2 = 0.05$.

e. Use (5.189) with

$$x = y_1 = Y(t_1),\ y = y_2 = Y(t_2),\ m_x = m_y = 0, \text{ and } \sigma_x^2 = \sigma_y^2 = 0.05$$

Also

$$\rho(\tau) = \frac{R_Y(\tau)}{R_Y(0)} = e^{-10|\tau|}$$

Set $\tau = 0.03$ to get $\rho(0.03) = 0.741$. Put these values into (5.189).

Problem 6.29

a. Choosing $f_0 = f_1$ moves $-f_1$ right to $f = 0$ and f_1 left to $f = 0$. The lowpass filtered version of the superposition of these two spectra is the power spectrum of $n_c(t)$ or $n_s(t)$. It is a v-shaped spectrum centered at $f = 0$ given by $S_{\text{LP}}(f) = \frac{1}{2}N_0 \left[1 - \Lambda\left(\frac{f}{f_2 - f_1}\right)\right]$

b. Choosing $f_0 = f_2$ moves $-f_2$ right to $f = 0$ and f_2 left to $f = 0$. Thus, the baseband spectrum can be written as $S_{\text{LP}}(f) = \frac{1}{2}N_0\Lambda\left(\frac{f}{f_2 - f_1}\right)$.

c. For this case both triangles (left and right) are centered around the origin and they add to give $S_{\text{LP}}(f) = \frac{1}{2}N_0\Pi\left(\frac{f}{f_2 - f_1}\right)$.

d. They are not uncorrelated for any case for an arbitrary delay. However, all cases give quadrature components that are uncorrelated at the same instant.

Problem 7.25

Solution: From the plot of the signal spectrum it is clear that:

$$k = \frac{A}{W^2}$$

Thus the signal power is: $S = 2\int_0^W \frac{A}{W^2} f^2 df = \frac{2}{3} AW$

The noise power is $N_0 B$. This yields the signal-to-noise ratio:

$$(SNR)_1 = \frac{2}{3}\frac{AW}{N_0 B}$$

If B is reduced to W, the SNR becomes:

$$(SNR)_2 = \frac{2}{3}\frac{A}{N_0}$$

This increases the signal-to-noise ratio by a factor of $\frac{B}{W}$.

6. Solution: $y(t) = 2n(t)\cos(2\pi f_c t + \theta)$

$$= 2(n_c(t)\cos 2\pi f_c t - n_s(t)\sin 2\pi f_c t)\cos(2\pi f_c t + \theta)$$

$$= n_c(t)\cos\theta + n_c(t)\cos(4\pi f_c t + \theta) - n_s(t)\sin\theta - n_s(t)\sin(4\pi f_c t + \theta)$$

$$= y_{lowpass}(t) + y_{bandpass}(t)$$

where $y_{lowpass}(t) = n_c(t)\cos\theta - n_s(t)\sin\theta$

$$y_{bandpass}(t) = n_c(t)\cos(4\pi f_c t + \theta) - n_s(t)\sin(4\pi f_c t + \theta)$$

$$E[y_{LP}(t)] = E[n_c(t)]\cos\theta - E[n_s(t)]\sin\theta = E[n(t)]\cos\theta - E[n(t)]\sin\theta = 0$$

$$E[y_{BP}(t)] = E[n_c(t)]\cos(4\pi f_c t + \theta) - E[n_s(t)]\sin(4\pi f_c t + \theta)$$ (assuming the noise has 0 mean)

$$= 0$$

$$E[y_{LP}^2(t)] = E[n_c^2(t)]\cos^2\theta + E[n_s^2(t)]\sin^2\theta - 2E[n_c(t)n_s(t)]\cos\theta\sin\theta$$

$$= E[n^2(t)] - 2\underbrace{E[n_c(t)n_s(t)]}_{=0 \text{ if } S_n(f) \text{ is symmetric around } f_c}\cos\theta\sin\theta$$

Similarly, $E[y_{BP}^2(t)] = E[n^2(t)] - 2E[n_c(t)n_s(t)]\cos(4\pi f_c t + \theta)\sin(4\pi f_c t + \theta)$