A Dynamic Auction for Spectrum Sharing

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Abstract— We study the design of a dynamic auction for sharing wireless spectrum between a primary high power user and one or more secondary low power users. In this market, the good being auctioned is transmission power which can be either allocated to the secondary users for transmission or bought by the primary user to reduce her interference. In this setting, the primary user may have a non-concave valuation, which prohibits applying standard designs of dynamic auctions. Moreover, a policy concern in such settings is often to prevent collusion and fraudulent bidding. Hence, Vickrey mechanisms may not provide the right incentives. We present a mechanism that selects a core outcome that minimizes the seller revenue. Such an allocation is efficient in equilibrium, limits the incentives to use shills, maximizes incentives for truthful bidding, and is incentive compatible when the Vickrey outcome is in the core.

I. INTRODUCTION

It has become widely recognized that the current approach for allocating wireless spectrum results in poor utilization of this resource. One way to improve on this is to allow for spectrum to be allocated on a finer scale both in time and space, e.g. by a "real-time spectrum market" [1]. Auctions are a natural approach for implementing such a market (e.g. [2], [3]). In this paper, we consider an auction design problem motivated by this application.

The prime objective of auction design is to construct a mechanism through which one or more goods are allocated between several agents in the presence of information asymmetry. Vickrey-Clarke Groves (VCG) theory shows that in such settings efficient outcomes can be obtained through a truthful mechanism. [4] show that this mechanism is essentially unique and the revenue equivalence theorem of [5] guarantees that, when allocating a single good, these merits do not come at the expense of the seller, as her revenue in a second price auction is equal to the expected revenue in a first price auction or in any other reasonable mechanism.

However, in multiple unit and combinatorial auctions, which naturally arise in spectrum sharing, the merits of VCG mechanisms may certainly come at the expense of the seller. By selling through an auction, a seller is committing to both allocations and payments dictated by the mechanism, regardless of the share of wealth (namely, the payments) that are transferred to the seller from the overall value that is created by this exchange. Indeed, in some cases the payments to the seller may be zero. This is further aggravated by the vulnerability of VCG mechanisms to various types of misrepresentation and collusion which all are at the expense of the seller. These create an incentive for the seller to deviate from the alleged dominant strategy equilibrium. Such deviations would typically increase the welfare of the seller, but decrease the overall efficiency. In the formal construction of the auction, this is, of course, impossible as the seller does not influence the outcome, but in practice the seller often has "outside" options to limit participation and nullify unfavorable outcomes. This is in particular true for wireless spectrum markets, where even in the most favorable circumstances there is a positive probability that technical issues may come up, and there is informational asymmetry that favors the seller as to the circumstances in which this happens. Moreover, given that a strategic seller may choose to play these outside strategies, other agents may have an incentive to deviate from their "inside" strategies. This shows that VCG mechanisms have an inherent instability that impact the outcome even when the inside mechanism has dominant strategies.[1]

One attempt to address these reservations would be to design auctions that would maximize revenue rather than social welfare. The revenue equivalence theory implies that for a single indivisible unit auction this would conflict with both efficiency and truthfulness. [5], [7] show that the same is true in the wider settings. Moreover, it would still leave in place the seller's outside options undermining stability. Our objective is therefore to find optimal *stable* allocation and payment schemes under these constraints.

More precisely, consider the class of mechanisms, in which each agent submits a valuation function. The seller can then choose a subset of agents that she would like to service, and price them at a level that is below their bid. In such a mechanism, the seller has the option to turn down all agents while the agents are committed to any price below their bid. The appropriate solution concept in this case is the *core*, the set of all outcomes, namely, allocations and payments, for which there is no welfare improving deviation by a coalition of agents

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[1]See [6] for a more detailed discussion of these shortcoming of VCG mechanisms.

including the seller.[2] In other words, the core consists of all the outcomes for which there is no coalition of bidders that could create more value than is suggested by the allocation. This in particular implies that the allocation is efficient, the bidders cannot benefit from shill bids or collusions, and the seller cannot benefit from excluding bidders (though she can of course allocate them nothing) [8].

The core in most cases is a large set and the strategic behavior of the agents depends on the way we choose a core allocation. Whenever the VCG outcome is in the core it can be shown that it is a core outcome that minimizes the revenue to the seller [8]. [8] and [9] also show that a mechanism that selects a core allocation of the *reported* valuations that minimizes seller revenue, has a full information equilibrium in the core of the *true* valuations, which is therefore efficient.

Our focus in this paper is on an auction for a single divisible good. In this setting, Ausubel's Clinching Mechanism [10] provides a dynamic auction which finds a core outcome that minimizes seller revenue when all agents have concave valuations. [3] Indeed, for that case the VCG outcome is always in the core and so this mechanism also finds the VCG outcome.

In the wireless spectrum market considered here one of the agents may not have a concave valuation, due to interference from the other users. In this case, our objective is to again construct a dynamic mechanism that finds a core outcome, which minimizes seller revenue. The mechanism we present will be a type of dynamic ascending auction that extends the Ausubel Clinching mechanism to the case where one bidder has a non-concave valuation.

A. Main Results

The non-concavity of an agent's utility or equivalently the complementarity of the commodity is a problem that has so far been largely ignored by the literature. Ausubel conditions his auction on concave utility and therefore cannot apply, moreover his auction uses one trajectory to discover market clearing price [10].

We construct a new auction, termed the *Fallback auction*, for multiple identical goods where one designated agent may have a non-concave valuation. It is based on Ausubel's ascending auction. Our main result is stated below.

Theorem 1: When all agents but one have concave utility functions, the Fallback auction has a full information equilibrium in the core.

B. Paper Structure

In section II we describe the specific setting which motivates our main result. Section III introduces the Fallback auction along with Ausubel's ascending auction. Examples of the Fallback auction follow in Section IV. In Section V we only

sketch the proof of Theorem 1 due to space limitation. Our result holds in both discrete and continuous time settings. The former requires additional tie breaking rules which add unnecessary difficulties to the analysis. We therefore do the analysis for a continuous time auction.

II. WIRELESS SPECTRUM MARKETS

We are interested in auctions for allocating wireless spectrum on a finer scale in both time and space than under traditional licensing. Such auctions would be run by a "spectrum manager" for a given frequency band/geographic area (see, e.g. [2], [3]). This managers task is essentially to allocate one or more wireless resources (e.g. bandwidth, power) among multiple users of the spectrum. Such resource allocation problems have been well-studied within the context of cellular systems in which the cellular operator is responsible for managing the resources. A key difference in spectrum markets is that the potential users of the spectrum are not all part of the same network and may have quite different requirements of the spectrum that are not known by the manager. For example there may be "large" primary user who requires high power to broadcast a signal to many users and "small" secondary users which only need to transmit a signal to a single nearby destination. To capture this we consider a model in which there is one primary agent and one or more secondary agents.[4]

The first question to ask in designing a spectrum market is "what is the good being allocated"? There are several possibilities here. For example, the manager could divide the band up into orthogonal frequency bands and auction each sub-band off to one agent. While this makes the allocation problem relatively straightforward, it puts sever restriction on frequency reuse. An alternative, which we adopt here, is for each user to spread its transmission across the entire band as in a code-division multiple access (CDMA) system. In this case the resource to be allocated is power and not bandwidth. This allows for frequency reuse, but also results in externalities among the users due to interference.

More precisely, we consider a model with n agents, where each agent is represented by a distinct transmitter-receiver pair. An example of two such agents is shown in 1. Let agent 1 be the primary agent so that agents $2, \ldots, n$ are the secondary agents. As in [3], we assume that the utility derived by each agent i is a function of the received *Signal-to-Interference and Noise Ratio (SINR)*

$$\gamma_i(\mathbf{q}) = \frac{q_i h_{ii}}{\sigma^2 + \sum_{j \neq i} q_j h_{ji}}, \qquad \text{for } i = 1, 2 \qquad (1)$$

where $\mathbf{q} = \{q_1\}$ is vector of transmission powers across all agents, σ^2 is the noise power and h_{ij} is the channel gain for transmitter i to receiver j. The externality that agent j causes

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[2]In our setting, any coalition to which goods are allocated must clearly include the seller. Hence, when we refer to a coalition of bidders, we implicitly assume that the seller is included.

[3]By a dynamic auction, we mean an auction in which the outcome is determined by a dynamic bidding process as in the traditional English (ascending bid) auction as opposed to a one-shot sealed bid auction. See [10] for a discussion of the advantages of such auctions.

[4]One possibility is that the primary agent has an exclusive license that prohibits other large users from using the spectrum but does allow smaller secondary agents. In our model if multiple primary agents where allowed then they could free ride on each other. In that case efficiency could only be obtained through additional contracting.
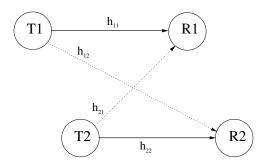
Fig. 1. Interference channel with two transmitter-receiver pairs

agent i is due to the interference term $q_j h_{ji}$ in this expression. Letting $U_i(\gamma_i)$ denote agent i's utility, an efficient allocation of power $\mathbf{q}$ maximizes $\sum_i U_i(\gamma_i)$.

We assume that the primary user transmits at a fixed power q_1 and is guaranteed that the interference from the secondary users does not exceed a given level q^{max}. The spectrum manager's task is then to allocate this total received power at the primary user's receiver i.e. the total power of the secondary users is constrained to satisfy $\sum_{i=2}^{n} q_i h_{i1} \leq q^{max}$. We also allow the primary use to participate in this auction. Each unit of power the primary user is allocated corresponds to a reduction in the total interference it will see, i.e. if it receives q_1^1 units of power, its total interference will be $q^{max} - q_1^1$. Let $x_1 = \frac{q_1^1}{q^{max}}$ and $x_i = \frac{q_i h_{i1}}{q^{max}}$, represent the normalized resources allocated to each agent, so that the resource constraint can be written as $\sum_{i=1}^{n} x_i \leq 1$.

The utility function $U_i(\gamma_i)$ is assumed to be a monotonic increasing concave function of γ_i for all i. However, it is more useful to consider this as a function of x_i. For each agent $i > 1$, we assume that the interference from the primary user is much larger than the interference from any secondary user, so that

$$\gamma_i(x_i) \approx \frac{x_i \frac{q_{max} h_{ii}}{h_{i1}}}{\sigma^2 + q_1 h_{1i}},$$

in which case clearly $U_i(x_i)$ will also be a monotonically increasing concave function of x_i.[5] On the other hand for Agent 1, we have

$$\gamma_1(x_1) = \frac{q_1 h_{11}}{\sigma^2 + q^{max}(1 - x_1)},$$

in which case $U_1(x_1)$ will again be an increasing function of x_1. However, $U_1(x_1)$ can be a convex function of x_1 if the following relation holds:

$$\frac{d^2 U_1(x_1)}{dx_1^2} = \frac{h_{11} q_1 (q^{max})^2}{(\sigma^2 + q^{max}(1 - x_1))^3} \left\{ \frac{d^2 U_1}{d\gamma_1^2} \gamma_1 + 2 \frac{dU_1}{d\gamma_1} \right\} > 0,$$
$$(2)$$

or equivalently if the coefficient of relative risk aversion, $-\frac{\gamma_1 U_1''(\gamma_1)}{U_1'(\gamma_1)}$, is less than 2 for all γ_1.

[5]If there is more than one secondary user and the interference from the other secondary users is not negligible, this once again users may free ride.

III. THE MODEL

We next give a model for an auction setting which abstracts the key features of the spectrum market described in the previous section. A seller wishes to allocate one unit of divisible good among n agents. Agent i obtains utility $U_i(x)$ from consuming $x \leq 1$ units of the good. Utilities are assumed to be private and quasi-linear with derivative u_i. When all U_i are concave, Ausubel's ascending auction can be used to allocate the good [10]. In this auction, sincere bidding is a dominant strategy that leads to an *ex post perfect equilibrium* and the VCG outcome is obtained. We describe this ascending auction in detail before we introduce our Fallback auction. For expositional convenience we assume the U_i's to be strictly concave.

A. Ausubel's Ascending Auction

If p is the per unit price of the good, strict concavity of $U_i(x)$ implies that

$$x_i(p) = \arg \max_{0 \leq x \leq 1} U_i(x) - p \cdot x \qquad (3)$$

is unique for any price p. Let $x_i(p)$ be agent i's demand at price p. Notice that $x_i(p)$ is continuous and strictly decreasing. In addition, $x_i(0) = 1$ and $x_i(p) \to 0$ as $p \to \infty$.

Initially the price is set to 0 and increased continuously. At each price p, each bidder i is asked to report her demand. We assume that each agent reports her demand $x_i(p)$ obtained from (3) truthfully. We show later that this truthful report is incentive compatible in Ausubel's ascending auction. Whenever the sum of demands from all agents is larger than supply (one unit of good), namely, $\sum_{i=1}^{n} x_i(p) > 1$, the auctioneer or the seller increases the price. The auction terminates when the market clearing price p^* is reached, at which p^* satisfies $\sum_{i=1}^{n} x_i(p^*) = 1$. Notice that the existence of such a price follows from the the fact that $x_i(p)$ for every agent is continuous and decreasing.

As the price increases from p to $p + \Delta p$, each agent i clinches, i.e., is allocated additional quantity $\Delta C_i(p) = C_i(p + \Delta p) - C_i(p)$ with the payment $p \cdot \Delta C_i(p)$ to the auctioneer. Here, $C_i(p) = [0, 1 - \sum_{j \neq i} x_j(p)]^+$ is the total clinched quantity of agent i at price p. If p^{-i} is the market clearing price when agent i is excluded, it is easy to see that $C_i(p) = 0$ for all $p \leq p^{-i}$. Hence, the total payment of agent i for the clinched quantity $C_i(p)$ up to the price $p \geq p^{-i}$ is the following:

$$P_i^{\text{AA}}(p) = \int_0^p \rho \frac{dC_i(\rho)}{d\rho} d\rho = - \int_{p^{-i}}^p \rho \frac{d(\sum_{j \neq i} x_j)}{d\rho} d\rho. \quad (4)$$

Notice also that $p^{-i} \leq p^*$ and $x_j(p^{-i}) \geq x_j(p^*)$ for all $j \neq i$.

When Ausubel's ascending auction ends with the market clearing price p^*, the final allocation of one unit of good among agents is socially optimal, namely maximizes sum of all agents' utilities. Now we show that the total payment $P_i'(p^*)$ of agent i after the auction ends is the same as the

VCG payment. The VCG payment for agent i is given by

$$P_i^{\text{VCG}} = \sum_{j \neq i} U_j(x_j(p^{-i})) - \sum_{j \neq i} U_j(x_j(p^*)). \quad (5)$$

Using information revealed from the agents during the auction and the first order condition $u_j(x_j(p)) = p$ for all j, the VCG payment becomes

$$
\begin{aligned}
P_i^{\text{VCG}} &= \sum_{j \neq i} \int_{x_j(p^*)}^{x_j(p^{-i})} u_j(x)dx & (6) \\
&= -\sum_{j \neq i} \int_{p^{-i}}^{p^*} u_j(x_j(\rho)) \frac{dx_j}{d\rho} d\rho & (7) \\
&= -\sum_{j \neq i} \int_{p^{-i}}^{p^*} \rho \frac{dx_j}{d\rho} d\rho & (8) \\
&= -\int_{p^{-i}}^{p^*} \rho \frac{d(\sum_{j \neq i} x_j)}{d\rho} d\rho, & (9)
\end{aligned}
$$

which is the same as the payment (4) at price p^*. Therefore, the ascending auction with concave utilities generates the VCG outcome. Moreover, the truthful report of demand $x_i(p)$ at the given price is incentive compatible for all agents [4].

So far we have not considered the payment of an agent appropriately in (3). When the auctioneer asks agent i to report her demand at the current price, we assume that each agent reports the $x_i(p)$ that maximizes $U_i(x) - p \cdot x$. However, this ignores the fact that agent i may already have clinched some quantities by the time the price in the auction has reached p. In fact, agent i has clinched $C_i(p) \geq 0$ with the payment $P_i'(p)$ at price p. She only pays for the additional amount of demand $x_i' - C_i(p)$ at the unit price p, where x_i' is agent i's demand that maximizes the following surplus $\pi_i(x, p)$:

$$
\begin{aligned}
\pi_i(x, p) &= U_i(x) - p \cdot [x - C_i(p), 0]^+ & (10) \\
&\quad - \int_0^p \rho \frac{dC_i(\rho)}{d\rho} d\rho.
\end{aligned}
$$

As we can see, $x_i'(p) = x_i(p)$. This is because the payment of agent does not depend on her bidding strategy (demand). This is the main reason that the truthful report of demand is incentive compatible in the ascending auction. Note that concavity of the utility function means that $x_i(p) \geq C_i(p)$ throughout the auction. The following algorithm 1 describes the ascending auction.

B. The Fallback Auction

The Fallback auction modifies Ausubel's ascending auction to account for the presence of a single agent with a non-concave utility function, henceforth called agent 1. All other agents have strictly concave utilities. In this case there may be no price p such that $\sum_{i=1}^n x_i(p) = 1$. The difficulty arises because $x_1(p)$ while non-increasing, may have discontinuities. Namely, there may be a p^t such that $\lim_{p \nearrow p^t} \sum_{i=1}^n x_i(p) \geq 1$ and $\sum_{i=1}^n x_i(p) < 1$ for all $p > p^t$. This means there is an excess supply when the auction reaches the price $p > p^t$. In addition, $x_1(p) = 0$ for all $p > p^t$ because of non-concavity

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Algorithm 1 Ausubel's Ascending Auction

Initialization:
$p \leftarrow 0$; $U_i, U_j^i \leftarrow 0$; $x_j, x_j^- \leftarrow 1$ for $i, j = 1, \ldots, n$
Dynamic:
while $\sum_{i=1}^n x_j > 1$ **do**
$\quad p \leftarrow p + \Delta p$
$\quad$ Ask each agent her demand x_i for price p.
$\quad x_i \leftarrow \max\{x_i, 1 - \sum_{j \neq i} x_j^-\}$; $x_i^- \leftarrow x_i$ for $i = 1, \ldots, n$
$\quad U_i \leftarrow U_i + p \cdot \Delta x_j$
$\quad$ **if** $\sum_{j \neq i} x_j = 1$ **then**
$\quad\quad U_j^i \leftarrow U_j$ for $j \neq i$
$\quad$ **end if**
end while
$P_i \leftarrow \sum_{j \neq i}(U_j - U_j^i)$ for $i = 1, \ldots, n$
Return $(x_1, \ldots, x_n)$ and $(P_1, \ldots, P_n)$

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of her utility function. Hence agent 1's demand at price that exceeds p^t can be less than what she has clinched. In Ausubel's ascending auction an agent is not allowed to relinquish her clinch. If agent 1's utility is convex, her surplus is forced to be zero or negative creating an incentive to deviate from sincere bidding.

The Fallback auction surmounts this difficulty by allowing agent 1 and only agent 1 to relinquish some of the units she has already clinched. If a Fallback price (defined below) is reached, we allow agent 1 to choose some smaller quantity clinched earlier and the auction terminates. The relinquished units from from agent 1 must then be reallocated to the other agents.

Definition 1: p^t is called a *fallback* price if $\lim_{p \nearrow p^t} \sum_{i=1}^n x_i(p) > 1$ and $x_1(p) = 0$ for all $p > p^t$.

To understand when a fallback occurs, suppose that agent 1 decides to fall back at some price p. At this price agent 1 is free to choose any quantity clinched earlier, i.e., $C_1(p')$ where $p' < p$. Agent 1's surplus from choosing $C_1(p')$ would be $U_1(C_1(p')) - \int_{p^{-1}}^{p'} \rho \frac{dC_1(\rho)}{d\rho} d\rho$. Clearly, she would choose the price p' that maximizes this surplus. This motivates the following definition.

Definition 2: The price p^* that solves

$$
\begin{aligned}
\max_{p^t \geq p \geq p^{-1}} & \pi_1(p) \\
&= \max_{p^t \geq p \geq p^{-1}} U_1(C_1(p)) - \int_{p^{-1}}^p \rho \frac{dC_1(\rho)}{d\rho} d\rho
\end{aligned}
$$

is called the *security* price.

The security price is the price that agent 1 chooses to maximize the surplus π_1 when she falls back. Therefore, the following equality holds at the fallback price:

$$\overline{\pi}_1(p^t) = \pi_1(p^*), \quad (11)$$

where $\overline{\pi}_1$ is the potential surplus agent 1 would receive if she was allocated her entire demand $x_1(p^t)$ (see the example in Section IV). Note that in the definition of the security price, the surplus maximization could be taken over the interval

$[p^{-1}, \infty)$. If the Fallback auction terminates at price p^* where $\sum_{i=1}^n x_i(p^*) = 1$, the resulting allocation and payments coincide with those of the ascending auction. In particular, the VCG outcome is obtained. If the Fallback auction terminates at a fallback price, namely $p^* = p^t$, then the auctioneer allocates the quantity $x_i(p^*)$ to each agent $i \geq 2$ and the amount $1 - \sum_{i \geq 2} x_i(p^*)$ to agent 1.

Once the auction ends, the payment that agent 1 makes for the allocation $C_1(p^*)$ is

$$P_1^{\text{FB}} = \int_0^{p^*} \rho \frac{dC_1(\rho)}{d\rho} d\rho. \quad (12)$$

To describe the payment for agent $i \geq 2$, we first modify the definition of p^{-i}. If the Fallback auction was applied only to the agents among $n \setminus i$ and terminates in a market clearing price, then p^{-i} is that market clearing price. Otherwise, it is the relevant security price. The payment that agent $i \geq 2$ makes is then

$$\begin{aligned} P_i^{\text{FB}} &= p^t \cdot \lim_{p \nearrow p^t} x_i(p) - \int_{p^*}^{p^t} \rho \frac{dx_i}{d\rho} d\rho \quad (13) \\ &= p^t \cdot \lim_{p \nearrow p^t} x_i(p) + U_i(x_i(p^*)) - U_i(x_i(p^t)). \end{aligned}$$

The payment consists of two terms. The first is the demand, charged at a rate of p^t per unit and the second is the increase in agent i's utility for the quantity re-allocated to her when agent 1 falls back. A pseudocode description of the complete Fallback auction is given in Algorithm 2..

IV. EXAMPLES

We explain the Fallback auction with the following example.

Example 1: Consider the allocation of one unit of a divisible good between three agents with the following utility functions:

$$\begin{aligned} U_1(x_1) &= \frac{1}{3}x_1^3 + \frac{11}{10}x_1, \quad (14) \\ U_2(x_2) &= 2x_2 - x_2^2, \quad (15) \\ U_3(x_3) &= 2x_3 - x_3^2. \quad (16) \end{aligned}$$

VCG Outcome: The efficient allocation is the solution to the following maximization problem:

$$\begin{aligned} \max \quad & \frac{1}{3}x_1^3 + \frac{11}{10}x_1 + 2x_2 - x_2^2 + 2x_3 - x_3^2 \quad (17) \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 1 \ ; \ x_1, x_2, x_3 \geq 0 \quad (18) \end{aligned}$$

Since agents 2 and 3 have identical concave utility, the solution should be $x_2 = x_3 = \frac{1-x}{2}$ and $x_1 = x$. The maximization problem is then equivalent to

$$\max_{0 \leq x \leq 1} \frac{1}{3}x^3 + \frac{11}{10}x + 2\left\{2(\frac{1-x}{2}) - (\frac{1-x}{2})^2\right\}, \quad (19)$$

which gives the optimal allocation $x_1 = 0.1127$ and $x_2 = x_3 = 0.44365$. Moreover, the VCG payments of all the agents can be calculated easily and are given by $P_1^{\text{VCG}} = 0.1191$ and $P_2^{\text{VCG}} = P_3^{\text{VCG}} = 0.6184$.

Fallback Outcome: Agent 1 has a convex utility function and her demand is $x_1(p) = 1$ until she falls back to her security

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Algorithm 2 Fallback Auction

Initialization:
fallback flag down ; $p \leftarrow 0$; $U_i, U_j^i \leftarrow 0$; $x_j, x_j^- \leftarrow 1$ for $i, j = 1, \ldots, n$
Dynamic:
while $\sum_{j=1}^n x_j > 1$ **and** fallback flag down **do**
  $p \leftarrow p + \Delta p$
  ask each agent her demand x_i for price p.
  if $x_1 > 1 - \sum_{j \neq 1} x_j^-$ **then**
    $x_i \leftarrow \max\{x_i, 1 - \sum_{i \neq j} x_j^-\}$; $x_i^- \leftarrow x_i$ for $i = 2, \ldots, n$
    $U_i \leftarrow U_i + p \cdot \Delta x_i$ for $i = 1, \ldots, n$
    if $\sum_{j \neq i} x_j = 1$ **then**
      $U_j^i \leftarrow U_j$ for $j \neq i$
    end if
  else
    fallback flag up
    $P_1 \leftarrow \sum_{j \neq 1}(U_j^* - U_j^i)$
    $P_i \leftarrow p \cdot x_i + U_i - U_i^*$ for $i = 2, \ldots, n$
    $x_i \leftarrow x_i^*$ for $i = 1, \ldots, n$
  end if
  ask agent 1 if p is secure.
  if 'yes' ($\iota = 1$) **then**
    $x_i^* \leftarrow x_i$ for $i = 1, \ldots, n$
    $U_i^* \leftarrow U_i$ for $i = 2, \ldots, n$
  end if
end while
if fallback flag down **then**
  $P_i \leftarrow \sum_{j \neq i}(U_j - U_j^i)$ for $i = 1, \ldots, n$
end if
Return $(x_1, \ldots, x_n)$ and $(P_1, \ldots, P_n)$

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clinch. Agents 2 and 3 demand the quantities for which their marginals exceed the price, or, $x_2(p) = x_3(p) = 1 - \frac{p}{2}$. If the price is $0 \leq p \leq 1$, sum of demands for any two agents exceeds the supply and therefore no agent clinches to anything. For $1 \leq p \leq 2$ the total demand of agents 2 and 3, $x_2(p) + x_3(p) = 2 - p$ drops below 1, and agent 1 begins to clinch $C_1(p) = p - 1$. This continues until agent 1 falls back or the price reaches $p = 2$ for which the demand of 2 and 3 drops to zero and the market clears. The clinch rate in this interval is constant, namely, $\frac{\partial C_1(\rho)}{\partial \rho} = 1$.

Assuming agent 1's demand ($x_1(p) = 1$) is satisfied without fallback, the potential surplus of agent 1 at price p is therefore

$$\begin{aligned} \bar{\pi}_1(p) &= U_1(x_1(p)) - p \cdot [x_1(p) - C_1(p)] \\ &\quad - \int_1^p \rho \frac{\partial C_1(\rho)}{\partial \rho} d\rho \quad (20) \\ &= \frac{43}{30} - p \cdot (2 - p) - \int_1^p \rho \, d\rho \quad (21) \\ &= \frac{1}{2}p^2 - 2p + \frac{29}{15}. \quad (22) \end{aligned}$$

On the other hand, the surplus of agent 1 given her actual clinch is given by

$$\pi_1(p) = U_1(C_1(p)) - \int_1^p \rho \frac{\partial C_1(\rho)}{\partial \rho} d\rho \tag{23}$$

$$= \frac{(p-1)^3}{3} + \frac{11}{10}(p-1) - \frac{p^2}{2} + \frac{1}{2}. \tag{24}$$

The security price is then obtained by

$$p^* = \arg\max\{\pi_1(p) : \text{s.t. } 1 \le p \le 2\} \tag{25}$$

$$= \frac{3 - \sqrt{0.6}}{2} = 1.1127. \tag{26}$$

The Fallback auction, therefore, has the following dynamics. As the price rises above $p = 1$, agent 1 begins to clinch. Her surplus increases until $p = p^*$, at which $\pi_1(p^*) = 0.0054$. As the auction continues, however, the security price of agent 1 remains $p^* = 1.1127$ since the surplus decreases with $p > p^*$. The price continues to increase until it reaches the fallback price p^t. At this price, $\overline{\pi}_1(p^t) = \pi_1(p^*)$, or

$$\frac{1}{2} \cdot (p^t)^2 - 2 \cdot p^t + \frac{29}{15} = \pi_1(p^*) = 0.0054, \tag{27}$$

which gives the fallback price $p^t = 1.6202$. Once agent 1 falls back to the allocation at the security price, the auction terminates and the final allocations among agents are the following:

$$x_1(p^*) = p^* - 1 = 0.1127, \tag{28}$$

$$x_2(p^*) = 1 - \frac{p^*}{2} = 0.4436, \tag{29}$$

$$x_3(p^*) = 1 - \frac{p^*}{2} = 0.4436. \tag{30}$$

At the fallback price, the demands of agent 2 and 3 are $x_2(p^t) = x_3(p^t) = 1 - \frac{p^t}{2} = 0.1899$ and the payments of the agents when the auction ends are then:

$$P_1^{\text{FB}} = \int_1^{p^*} \rho \frac{\partial C_1(\rho)}{\partial \rho} d\rho = 0.1190, \tag{31}$$

$$P_2^{\text{FB}} = 2 x_2(p^*) - (x_2(p^*))^2 - 2 x_2(p^t)$$
$$+ (x_2(p^t))^2 + p^t \cdot x_2(p^t) = 0.6543, \tag{32}$$

$$P_3^{\text{FB}} = 0.6543. \tag{33}$$

V. FALLBACK AUCTION AS A CORE-SELECTING AUCTION

As shown in Section IV, the final allocations of the Fallback auction are the same as the VCG allocations, which maximizes the sum utilities of the agents. This is only possible when agents reveal their demand truthfully. However, the payment of every agent except agent 1 is not the VCG payment, and therefore sincere bidding (truthful reporting) is not a dominant strategy. Instead, we prove that the Fallback auction with truthful bidding leads to a core allocation with minimum total payment to the seller.

We repeat some of concepts from [8] since we heavily rely on those to prove Theorem 1 in the Fallback auction. For any coalition S, an assignment $\hat{x}$ is feasible for coalition S, written $x \in F(S)$, if (1) $\sum_i x_i \le 1$ and (2) for all j, if $j \notin S$ or the seller is not in S, then $x_j = \varnothing$. That is, a bidder can have a non-null assignment when coalition S forms only if that bidder and the seller are both in the coalition. The coalition value function or characteristic function is defined by

$$w_U(S) = \max_{x \in F(S)} \sum_{j \in S} U_j(x_j). \tag{34}$$

If the payment of the auction P_j is made to the seller by each agent j, then the associated payoffs are given by $\sum_{j=1}^n P_j$ for the seller and $\pi_j = U_j(x_j) - P_j$ for each bidder j. The payoff profile is individually rational if $\pi_j \ge 0$ for all j. An *imputation* is a feasible, non-negative payoff profile. An imputation is in the core if it is efficient and unblocked:

$$Core(n, w) = \left\{ \pi \ge 0 \,\Big|\, \sum_{j \in n} \pi_j = w(n) \right.$$

$$\left. \text{and } (\forall S \subseteq n) \sum_{j \in S} \pi_j \ge w(S) \right\}. \tag{35}$$

Now we consider the Fallback auction. If the Fallback auction (i.e. Algorithm 2) terminates with the fallback flag down, it is equivalent to Ausubel's Ascending auction. We therefore assume the algorithm terminates with the fallback flag up, and prove that we obtain an equilibrium outcome in the core. This is done in several steps. First, we show that the resulting allocation of the Fallback auction is efficient under truthful bidding. Second, for two agents, the VCG outcome is obtained. Especially, the payments of all agents are the same as the VCG payments. From a standard argument (see for example [6] or [11]) it follows that bidding truthfully is an ex-post perfect equilibrium. Finally, we show that for $n > 2$ agents, the truthful bidding outcome of the Fallback auction lies in the core with the seller's minimum payoff. From [12], the bidding strategies of the agents according to a *truncation profile* is a full information equilibrium.

Lemma 2: Assuming truthful bidding among agents, the allocation of the Fallback auction is efficient.

Proof: We show that $x_2(p^*), \ldots, x_n(p^*)$ and $x_1(p^*) = 1 - \sum_{i \ge 2} x_i(p^*)$ are solutions to the following maximization problem Π.

$$\max \quad U_1(x_1) + \sum_{i=2}^n U_i(x_i) \tag{36}$$

$$\text{s.t.} \quad \sum_{i=1}^n x_i = 1$$

$$x_i \ge 0 \; \forall i \in n.$$

For given $Q \in [0, 1]$, consider the following problem.

$$F(Q) = \max \sum_{i=2}^n U_i(x_i) \tag{37}$$

$$\text{s.t.} \quad \sum_{i=2}^n x_i = Q$$

$$x_i \ge 0 \; \forall i \in n \setminus \{1\}.$$

There is a Lagrange multiplier p that corresponds to the market clearing price such that Q units are distributed among the agents $n \setminus \{1\}$. Therefore, $F(Q) = \sum_{i=2}^{n} U_i(x_i(p))$. For example, when $Q = 1$, the corresponding Lagrange multiplier is p^{-1}. Now the problem Π can be reformulated as

$$\max_{p \geq p^{-1}} U_1(1 - \sum_{i \geq 2} x_i(p)) + \sum_{i \geq 2} U_i(x_i(p)). \tag{38}$$

The objective function can be expressed as

$$U_1(1 - \sum_{i \geq 2} x_i(p)) + \sum_{i \geq 2} \int_{p^{-1}}^{p} u_i(x_i(\rho)) \frac{dx_i(\rho)}{d\rho} d\rho$$

$$= U_1(C_1(p)) - \int_{p^{-1}}^{p} \rho \frac{dC_1(\rho)}{d\rho} d\rho, \tag{39}$$

and this is maximized by the security price p^* (See Definition 2). ∎

Lemma 3: Suppose $n = 2$ and agent 1 has an increasing convex utility function. Then truthful bidding is an *ex post perfect equilibrium* of the Fallback auction that charges the VCG payment to each agent.

Proof: When the fallback flag is up, the payment of agent 1 is

$$P_1^{\text{FB}} = \int_0^{p^*} \rho \frac{dC_1(\rho)}{d\rho} d\rho = \int_{p^{-1}}^{p^*} \rho \frac{dC_1(\rho)}{d\rho} d\rho \tag{40}$$

$$= U_2(x_2(p^{-1})) - U_2(x_2(p^*)), \tag{41}$$

which is by definition the VCG payment. Moreover, the payment of agent 2 is given by equation (13):

$$P_2^{\text{FB}} = p^t \cdot \lim_{p \nearrow p^t} x_2(p) + U_2(x_2(p^*)) - U_2(x_2(p^t)). \tag{42}$$

From the definition of the fallback price, $\overline{\pi}_1(x_1(p^t), p^t) = \pi_1(x_1(p^*), p^*)$. Namely,

$$U_1(1) - p^t \cdot (1 - C_1(p^t)) - \int_0^{p^t} \rho \frac{dC_1(\rho)}{d\rho} d\rho \tag{43}$$

$$= U_1(C_1(p^*)) - \int_{p^{-1}}^{p^*} \rho \frac{dC_1(\rho)}{d\rho} d\rho, \tag{44}$$

or

$$U_1(1) - p^t \cdot x_2(p^t) = U_1(x_1(p^*)) + \int_{p^*}^{p^t} \rho \frac{dC_1(\rho)}{d\rho} d\rho \tag{45}$$

$$= U_1(x_1(p^*)) + U_2(x_2(p^*)) - U_2(x_2(p^t)). \tag{46}$$

Here, we use the following facts in the auction: $x_1(p^t) = 1$ and $C_1(p^t) = 1 - x_2(p^t)$. Therefore, the payment of agent 2 becomes

$$P_2^{\text{FB}} = U_1(1) - U_1(x_1(p^*)) = U_1(x_1(p^{-2})) - U_1(x_1(p^*)), \tag{47}$$

and this is exactly the VCG payment of agent 2. For $n = 2$, the Fallback auction with truthful bidding reaches the efficient allocation with the VCG payment. From [4], any incentive compatible, individually rational and efficient mechanism must charge VCG payments. Therefore, for $n = 2$, truthful bidding

is an *ex post perfect equilibrium* of the Fallback auction that returns the VCG outcome. ∎

In [8], Day and Milgrom argue that core-selecting auctions that minimize the seller's payoff maximize incentives for truthful reporting and they produce the Vickrey outcome when it lies in the core. Theorem 3 in [8], especially, shows that a truncation report is a full information equilibrium. For a given $\alpha \geq 0$, a truncation report for agent i, corresponds to that agent reporting an α-truncation of her true utility, i.e. $U_i(x_i) - \alpha$. From the following Lemma, we show that the Fallback auction minimizes the seller's payoff and the payoffs of all agents lies in the core.

Lemma 4: For $n > 2$, if all agents bid truthfully, the Fallback auction finds an imputation in the core with minimum payoff of the seller.

We omit the proof of Lemma 4 here due to space limitation. The bidding strategies of the agents according to the profile of π_i truncations of $U_i(x_i)$ for all i is, therefore, a full information equilibrium in the Fallback auction and this leads to the efficient outcome.

Corollary 5: The bidding strategies of the agents according to the profile of π_i truncations of $U_i(x_i)$ for all i is a full information equilibrium in the Fallback auction.

Proof: The allocation of the Fallback auction with truthful bidding is a bidder optimal allocation according to [8]. Therefore, the bidding strategies of the agents according to the profile of π_i truncations of $U_i(x_i)$ or $\widehat{U}(x_i) = U_i(x_i) - \pi_i$ is a full information equilibrium in the Fallback auction (See Theorem 3 in [8]). The only part left for the proof of Corollary 5 is whether the Fallback auction with $\widehat{U}_i(x_i)$ gives the same allocation as that with $U_i(x_i)$. As the unit price goes up from 0, each agent i asks for the quantity $x_i = \arg\max_{0 \leq x_i \leq 1} \widehat{U}_i(x_i) - p \cdot x_i = \arg\max_{0 \leq x_i \leq 1} U_i(x_i) - \pi_i - p \cdot x_i$. Therefore, each concave agent asks for the exact same quantity as if it responds according to it's true utility except that at a certain price $\widetilde{p}$, it's demand suddenly becomes zero. We also need to show that $\widetilde{p} \geq p^t \geq p^*$ for all concave agents, which can be easily done since $\pi_i = U_i(x_i(p^t)) - p^t \cdot x_i(p^t)$ for $\forall i \in n$. ∎

VI. CONCLUSIONS AND OPEN PROBLEMS

We studied an auction model motivated by spectrum sharing in which there is one bidder with a non-concave valuation and $n - 1$ bidders with concave valuations. For this setting we presented the Fallback auction, a dynamic auction which has a full information equilibrium in the core. For $n = 2$ agents this is also the VCG outcome; for $n > 2$ agents it is the core allocation which minimizes the seller's revenue. This auction dynamically elicits information from the agents to determine an efficient outcome. It would be of interest to determine if this is the minimal information that must be elicited for obtaining an such an outcome. Though in the spectrum sharing model we presented only one agent has a non-concave valuation; it would also be of interest for other applications to extend these ideas to models with more than one non-concave valuations.

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