EECS 310: Discrete Math **Lecture 10**
Counting

Reading: MIT OpenCourseWare 6.042 Chapter 9.1, 9.3-9.5

Review

Note: Graph theory exam covers lectures 5-9, chapter 5 of book.

Concepts:

- general: (undirected simple) graphs, degree, connectivity, connected components, subgraphs, isomorphism
- types of graphs: paths, cycles, trees, complete graphs (cliques), bipartite graphs, planar graphs
- matching: perfect matchings in bipartite graphs, stable matchings
- paths/cycles: counting with adjacency matrix, Eulerian, Hamiltonian
- coloring: chromatic number
- planarity: planar embeddings

Theorems:

- Sum of degrees is twice number of edges.
- Hall's theorem: there's a matching that covers L iff $\forall S \subset L, |S| \leq |N(S)|$.
- Stable marriage, men-proposing is man-optimal.
- Graph is Eulerian iff all vertices have even degree.
- Chromatic number is at most max degree plus one, at least max degree.
- Planar graphs: $n - m + f = 2$ and $m \leq 3n - 6$
- Equivalent definitions of trees.

[[*Knowing how the proofs work will help you.*]]

Computing Sums

Question: # of nodes in a binary tree?

Question: \$1M today or \$50,000 for 20 yrs? for the rest of your life? forever?

Some known sums:

- linear: $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$
- geometric: $\sum_{i=0}^{n-1} x^i = \frac{1-x^n}{1-x}$

Value of money: at interest rate of 8% per year,

Future worth of \$10 today

- $(1 + 0.08) \cdot 10 = 10.80$ in 1 year
- $(1 + 0.08)^2 \cdot 10 = 11.66$ in 2 years

Today's worth of \$10 in

- 1 year: $10/(1+0.08) = 9.26$ today
- 2 years: $10/(1+0.08)^2 = 8.57$ today

Fact: n-year m-payment annuity with interest rate p is worth

$$\begin{aligned} V &= \sum_{i=1}^{n} m \cdot \left(\frac{1}{1+p}\right)^{i-1} \\ &= m \sum_{j=0}^{n-1} \left(\frac{1}{1+p}\right)^{j} \\ &= m \sum_{j=0}^{n-1} x^j \end{aligned}$$

Question: How to solve sum?

Perturbation method

linear

$$S = 1 + 2 + \ldots + n$$

$$S = n + (n-1) + \ldots + 1$$

summing, get

$$2S = (n+1) + (n+1) + \ldots + (n+1) = n(n+1)$$

geometric

$$S = 1 + x + \ldots + x^{n-1}$$

$$xS = x + x^2 + \ldots + x^n$$

subtracting, get

$$(1-x)S = 1 - x^n$$

Note: more examples, see generating functions (Chapter 12) [[*very useful but we probably won't get to it, highly recommended reading*]]

Fact: \$50,000 for 20 years is worth \$530,180. [[*Million dollar lottery only worth half a million!*]]

Fact: \$50,000 for rest of your life is worth at most \$675,000 even if you live forever!

infinite geometric

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} x^i = \lim_{n\to\infty} \frac{1-x^n}{1-x} = \frac{1}{1-x}$$

for $|x| < 1$.

examples:

- $1 + 2 + 4 + \ldots + 2^{n-1}$
$$= \sum i = 0^{n-1} 2^i = \frac{1-2^n}{1-2} = 2^n$$
- $1 + 1/2 + 1/4 + \ldots$
$$= \sum_{i=0}^{\infty} (1/2)^i = \frac{1}{1-1/2} = 2$$
- $1 - 1/2 + 1/4 - \ldots$
$$= \sum_{i=0}^{\infty} (-1/2)^i = \frac{1}{1+1/2} = 2/3$$
- $0.99999\ldots$
$$= 0.9 \sum_{i=0}^{\infty} (1/10)^i = 0.9 \cdot \frac{1}{1-1/10} = 1$$

Note: Geometric sum approximately equal to largest term!

Differentiation Method

Question: \$1000 now or \$$5i$ on i'th birthday forever?

$$V = \sum_{i=0}^{\infty} \frac{im}{(1+p)^i} = m \sum_{i=0}^{\infty} i x^i$$

Claim: $\sum_{i=0}^{n-1} ix^i = \frac{x-nx^n+(n-1)x^{n+1}}{(1-x)^2}$

Proof: Differentiate geometric sum.

$$\begin{aligned} \frac{d}{dx}\sum_{i=0}^{n-1} x^i &= \sum_{i=0}^{n-1} \frac{d}{dx} x^i \\ &= \sum_{i=0}^{n-1} ix^{i-1} \end{aligned}$$

so

$$\begin{aligned} \sum_{i=0}^{n-1} ix^i &= x \cdot \left(\sum_{i=0}^{n-1} ix^{i-1} \right) \\ &= x \cdot \left(\frac{d}{dx} \sum_{i=0}^{n-1} x^i \right) \\ &= x \cdot \left(\frac{d}{dx} \left(\frac{1-x^n}{1-x} \right) \right) \end{aligned}$$

Claim: For $|x| < 1$, $\sum_{i=0}^{\infty} ix^i = \frac{x}{(1-x)^2}$.

Note: So $5 forever is worth only $844!

[[*Surprising that it's finite, but note that geometric growth is much stronger than linear growth, so geometric decrease wipes out linear increase.*]]

Approximating Sums

Integration Method

Replace sum by integral and add in first or last term of sum.

Claim: Let $f(.)$ be non-decreasing continuous function and let

$$S = \sum_{i=1}^{n} f(i)$$

and

$$I = \int_1^n f(x)dx.$$

Then

$$I + f(1) \le S \le I + f(n).$$

Similarly if $f(.)$ is non-increasing, then

$$I + f(n) \le S \le I + f(1).$$

Proof:

- Draw step function, height at i is $f(i)$.
- Note area of curve under step function is $\sum_{i=1}^{n} f(i)$.
- Draw continuous curve $f(x)$ from 1 to n.
- Note under step function, hits at left-corners.
- Note area under curve is integral.
- For lower bound, can add back in left-most step.
- For upper bound, shift left one, add back in right-most step.

Example: Harmonic numbers $H_n = \sum_{i=1}^{n} \frac{1}{i}$

Draw picture:

$$\frac{1}{n} + \int_0^n \frac{1}{x} dx \le \sum_{i=1}^{n} \frac{1}{i} \le 1 + \int_1^n \frac{1}{x} dx,$$

so by integration method,

$$\frac{1}{n} + \ln(n) \le \sum_{i=1}^{n} \frac{1}{i} \le 1 + \ln(n).$$

Note: H_n is *very close* to $\ln(n)$ (off by a small constant).

Def: asymptotic equality: $f(x) \sim g(x)$ iff $\lim_{x\to\infty} f(x)/g(x) = 1$

So $H_n \sim \ln(n) + c$ for constant c.

Double Sums

For $0 \leq x, y < 1$

$$\begin{aligned}
\sum_{n=0}^{\infty} \sum_{i=0}^{n-1} y^n x^i &= \sum_{n=0}^{\infty} y^n \sum i = 0^{n-1} x^i \\
&= \sum_{n=0}^{\infty} y^n \frac{1 - x^n}{1 - x} \\
&= \frac{1}{1-x} \sum_{n=0}^{\infty} y^n - \frac{1}{1-x} \sum_{n=0}^{\infty} (xy)^n \\
&= \frac{1}{(1-x)(1-y)} - \frac{1}{(1-x)(1-xy)}
\end{aligned}$$

etc.

What if inner sum has no closed form?

Changing order of summation

Example: Sum of Harmonic numbers

$$\sum_{i=1}^{n} \sum_{j=1}^{i} \frac{1}{j}$$

First attempt:

- inner sum $H_i \sim \ln(i)$
- use integration method

$$\int_1^n \ln(x) dx = (x \ln(x) - x)|_1^n$$
$$= n \ln n - n + 1$$

Exact answer?

- write pairs (i, j) in summation in table
- given sum: adds entries in each row, then adds row sums
- instead: sum entries in each column, then add column sums

$$\begin{aligned}
\sum_{i=1}^{n} \sum_{j=1}^{i} \frac{1}{j} &= \sum_{j=1}^{n} \sum_{i=j}^{n} \frac{1}{j} \\
&= \sum_{j=1}^{n} \frac{1}{j} \sum_{i=j}^{n} 1 \\
&= \sum_{j=1}^{n} \frac{1}{j} (n - j + 1) \\
&= (n+1) \sum_{j=1}^{n} \frac{1}{j} - \sum_{j=1}^{n} 1 \\
&= (n+1) H_n - n
\end{aligned}$$