EECS 310: Discrete Math	**Lecture 11**
Counting	

Reading: MIT OpenCourseWare 6.042 Chapter 11

Counting

Bijection Rule

Def: bijection $f : A \to B$ is perfect matching of $a \in A$ and $b \in B$

Claim: if exists bijection $f : A \to B$, then $|A| = |B|$

Example: doughnuts

- A = ways to select dozen doughnuts from 5 varieties
- B = number 16-bit strings with exactly 4 ones

representation:

- element of A:

  2 choc., 0 lemon, 5 sugar, 2 glazed, 2 plain

  $$00 - \ldots - 00000 - 00 - 00$$

  choc.—lemon—sugar—glazed—plain

- element of B:

  replace "–" with 1

bijection:

- dozen doughnuts

  c choc., l lemon, s sugar, g glazed, p plain

- bit string

  c 0's, 1, l 0's, 1, s 0's, 1, g 0's, 1, p 0's

[[*How to count bit-strings? or other sequences?*]]

Sum/Product Rule

A = set of cakes

B = set of pies

Question: How many ways are there to pick one cake and one pie?

$|A| \times |B|$

Question: How many ways to pick one dessert?

$|A| + |B|$

Question: How many ways are there to pick one cake and one pie if some cakes are pies? [[*Boston creme pie*]]

Draw Venn diagram – $|A \cup B| = |A| + |B| - |A \cap B|$

In general, given finite sets $A_1, \ldots, A_n$,

- Product Rule: There are $\prod_{i=1}^{n} |A_i|$ ways to select n elements, one element from each set.
- Inclusion/Exclusion: There are $| \cup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i| - \sum_{i=1}^{n} |A_i \cap A_j| + \ldots +$

$(-1)^{n+1}|A_1 \cap \ldots \cap A_n|$ ways to select one element.

Example: Sum/Product Rule:

- How many two-digit numbers are there?

  9 choices for 1st digit $\times 10$ for 2nd $= 90$

- If $|A| = n$ and $|B| = m$, how many functions are there from A to B?

  n choices for 1st elt $\times \ldots \times$ n for m'th elt $= m^n$

- How many subsets of an n-element set?

  bijection to binary strings: subset S maps to string s with i'th bit 1 iff $i \in S$

  2 choices for each of n bits $= 2^n$

- How many passwords consisting of letters and digits that
  - start with a letter
  - have length 6-8

  Let F be set of first symbol, S_5, S_6, S_7 be strings of $5, 6, 7$ symbols

  Password is one elt of F appended with one elt of S_5 or S_6 or S_7

$$(F \times S_5) \cup (F \times S_6) \cup (F \times S_7)$$

$$= |F \times S_5| + |F \times S_6| + |F \times S_7|$$

$$= |F| \cdot |S_5| + |F| \cdot |S_6| + |F| \cdot |S_7|$$

$$= 52 \cdot 62^5 + 52 \cdot 62^6 + 52 \cdot 62^7$$

1.8×10^14

Example: Inclusion/Exclusion, Sieve of Eratosthenes (200 B.C.)

How many primes < 100?

- Count composites: have a prime divisor ≤ 10.
- Primes ≤ 10 are $2, 3, 5, 7$.
- Let
  - $A_2 = \{n \leq 100 : 2|n\}$
  - $A_3 = \{n \leq 100 : 3|n\}$
  - $A_5 = \{n \leq 100 : 5|n\}$
  - $A_7 = \{n \leq 100 : 7|n\}$
- # composites $= |A_2 \cup A_3 \cup A_5 \cup A_7| - 4$

  (for primes $2, 3, 5, 7$)
- $|A_p| = \lfloor \frac{100}{p} \rfloor$, $|A_{p,q}| = \lfloor \frac{100}{pq} \rfloor$, etc.

  $|A_2 \cup A_3 \cup A_5 \cup A_7| = \ldots = 78$
- # primes $= 99 - (78 - 4) = 99 - 74 = 25$.

Sieve crosses out numbers divisible by 2 and thereafter numbers divsible by first on list.

Permutations

Generalized product rule: sequences of r elts of n-elt set is

$$n(n-1)(n-2)\ldots(n-r+1)$$

Example: Race with n horses, how many options for win, place, show?

By product rule, $n(n-1)(n-2)$.

Example: Chess problem

ways place white rook, black rook, neither attacks other

- (c_w, r_w) = col/row of white rook
- (c_b, r_b) = col/row of black rook

- board position is 4-digit sequence (c_w, r_w, c_b, r_b)

  e.g., $(1,1,8,8)$ or $(8,8,1,1)$
- c_w, r_w have 8 choices each
- given (c_w, r_w), c_b, r_b have 7 choices each

is 56^2.

Def: *permutation* $\sigma(n)$ of n distinct objects is an ordered selection.

Claim: # permutations of n elements is

$$n! = n(n-1)\dots 1.$$

[[*proof - product rule*]]

Division Rule

Claim: If $f : A \to B$ maps exactly k elts of A to each elt of B, then $|A| = k|B|$.

Example: Chess problem

ways place two white rooks in diff row/col

- (c_1, r_2) = col/row of first rook
- (c_2, r_2) = col/row of second rook
- board position is 4-digit sequence (c_1, r_1, c_2, r_2)

  but $(1,1,8,8) = (8,8,1,1)$
- so divide by 2

is $56^2/2$.

Combinations

Example: pizza toppings

pizzas with 2 toppings

- pepper
- onion
- mushroom

Product+division rule:

- (t_1, t_2) sequence of toppings
- $(t_1, t_2) = (t_2, t_1)$ so divide by 2
- $3 \times 2 = 6$ sequences
- so 3 ways

Def: *combination* is unordered selection of r objects from n objects

Claim: # combinations of r from n is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Proof: bijection

- take permutation of n
- let first r be selection
- fix permutation:
  - any other permutation with same first r gives same unordered selection
  - order of first r and last $n-r$ doesn't matter
  - $r!$ perms of first r, $(n-r)!$ perms of rest
- $r!(n-r)!$ ways to unorder them

□

Example: # n-bit seq with exactly k ones [[*recall came up in doughnut problem*]] select positions of 1's: $\binom{n}{k}$

Example: # ternary strings with k_1 1's and k_2 2's?

Example: Rearrangements of BOOKKEEPER

- pretend distinct: 10! ways
- fix over-counting: divide by
  - 2! (switch O's)
  - 2! (switch K's)
  - 3! (reorder E's)

Example: # walks 5 blocks in each direction [[*so north 2 blocks, west one block, south two blocks, north two blocks, etc.*]] directions

- sequences of 5 N's, 5 W's, 5 E's, 5 S's
- so: $20!/(5!)^4$

Bars and Stars

[[*Important that all elements in set and in selection are distinct. Finding permutations/combinations with repitition is slightly harder.*]]

Question: How many ways to give n candycorn to k kids so each kid gets at least one piece of candy?

- Stars = candy, draw on line
- Bars = bucket boundaries, draw between stars
- $n-1$ places for bucket boundaries; $k-1$ boundaries
- Answer: $\binom{n-1}{k-1}$

Question: How many ways if some kids can get no candy?

Claim: Answer: $\binom{n+k-1}{k-1}$

Proof:

1. Placing n objects in k bins allowing empty bins is like placing $n+k$ objects in k bins disallowing empty bins
2. Have $n+k-1$ symbols of which $k-1$ must be bars

□

[[*Note above two are consistent; to give each kid at least one candycorn, could have taken k candycorn out of the n and then done the second approach. Get same answer.*]]

Example:

- How many *positive* solutions to $a+b=10$?
- How many *non-negative* solutions to $a+b=10$?

Identities

What is $\sum_{r=0}^{n}\binom{n}{r}\binom{2n}{n-r}$?

Often have a nice combinatorial proof.

Some easy ones:

- $\binom{n}{r}=\binom{n}{n-r}$: choose elements to keep or to throw away.
- $\sum_{k=0}^{n}\binom{n}{k}=2^n$: all subsets of an n-element set.
- $(x+y)^n=\sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^j$ (the binomial theorem): coeff of $x^{n-j}y^j$, must choose j terms in product from which to take y.

- $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ (Pascal's triangle): remove arbitrary element, either this is in the subset of size k (first term) or not (second term).

Claim: $\sum_{r=0}^{n} \binom{n}{r}\binom{2n}{n-r} = \binom{3n}{n}$

Proof: Let S be deck of cards with n red cards and $2n$ black cards.

- RHS: $\binom{3n}{n}$ is the number of n-card hands.
- LHS:
  - $\binom{n}{r}$ is the number of way to pick r red cards
  - $\binom{2n}{n-r}$ is the number of ways to pick $n - r$ black cards

  so $\binom{n}{r}\binom{2n}{n-r}$ is number of ways to pick a hand with exactly r red cards.

Result follows by summing over r. □

Poker Hands

52 cards in a deck:

- 4 suits: Spades, Hearts, Diamonds, Clubs
- 13 values: $1, \ldots, 10$, Jack, Queen, King, Ace

5-card draw: each player given 5 cards

$$\binom{52}{5} = 2,598,960$$

possible hands.

How many ways to get:

- Four-of-a-kind – 4 cards with same *value*

  $$\{2H, 2S, 2D, 2C, 5S\}$$

  Hand fully specified by sequence specifying
  - value of 4 cards
  - value of extra card
  - suit of extra card

  Count ways to pick sequence
  - 13 choices for value of 4 cards
  - 12 choices for value of extra card
  - 4 choices for suit of extra card

  so $13 \cdot 12 \cdot 4 = 624$ four-of-a-kinds, or about 1 in 4000 hands.

- Full house – 3 cards of one value, 2 cards of another value

  $$\{7H, 7S, 7D, JH, JS\}$$

  Sequence
  - value of triple – 13
  - suits of triple – $\binom{4}{3}$
  - value of pair – 12
  - suits of pair – $\binom{4}{2}$

  so $13 \cdot 4 \cdot 12 \cdot 6 = 3744$, or about 1 in 700 hands.

- Two pair – 2 cards of one value, 2 cards of another value

  $$\{4S, 4H, 6D, 6H, KC\}$$

  Sequence
  - value of first pair – 13

- suits of first pair – $\binom{4}{2}$
- value of second pair – 12
- suits of second pair – $\binom{4}{2}$
- value of extra card – 11
- suit of extra card – 4

WRONG ANSWER!, each hand gives rise to two distinct sequences:

$$(4, \{S, H\}, 6, \{D, H\}, K, C)$$
$$= (6, \{D, H\}, 4, \{S, H\}, K, C)$$

Solution: mapping is 2-to-1, so divide by 2 (*division rule*)

Number of two-pair hands is: $13 \cdot 6 \cdot 12 \cdot 6 \cdot 11 \cdot 4/2 = 123,552$, or about 1 in 20.

Note: alternatively, could come up with different sequence to count, e.g.,

- values of two pairs – $\binom{13}{2}$
- suits of lower-valued pair – $\binom{4}{2}$
- suits of higher-valued pair – $\binom{4}{2}$
- value of extra card – 11
- suit of extra card – 4

- Hands with every suit

  $\{AS, AH, JD, 4C, 10H\}$

  Sequence

  - value of spade – 13
  - value of heart – 13
  - value of diamond – 13
  - value of club – 13
  - suit of extra card – 4
  - value of extra card – 12

  Overcount?

  $$(A, A, J, 4, H, 10) = (A, 10, J, 4, H, A)$$

  Must divide by 2.

Pigeon-hole

Pigeon-hole principle: If n objects are placed into r boxes, then at least $\lceil n/r \rceil$ must be in the same box.

Claim: Consider numbers $\{1, 2, \ldots, 2n\}$. Then in any set A of size $n+1$, at least two elts are relatively prime.

Proof:

- Pigeons – elts of A
- Holes – hole i is for pigeons between i and $i+1$

That is, at least two numbers are 1 apart and hence relatively prime. □

Claim: Consider numbers $\{1, 2, \ldots, 2n\}$. Then in any set A of size $n+1$, there are always two numbers such that one divides the other.

Proof: For each $a \in A$, write $a = 2^k m$, where m is odd and $1 \le m \le 2n-1$.

- Pigeons – elts of A
- Holes – odd numbers m

By pigeon-hole, must be two numbers a and a' with same hole m. That is, $a = 2^k m$ and $a' = 2^{k'} m$, so one is a multiple of the other. □

Claim: Five cities on an alien planet, at least 4 are in same hemisphere.

Proof: Draw great circle through two of five points, by pigeonhole two of remaining three cities are in same hemisphere. □

Magic Trick

[[*Ok to skip.*]]

- Audience: choose 5 cards
- Assistant: show magician 4 cards, one at a time
- Magician: announce missing card

Counting:

- Audience: $\binom{52}{5}$
- Assistant: $4! = 24$ ways to show 4 of five cards

but ... 48 possible cards, so not enough permutations to narrow it down.

Idea: Assistant gets to pick *order* and also *which card to leave out!*.

Create bipartite graph, find matching:

- LHS: all audience choices
- RHS: all sequences of 4 distinct cards
- edges: if sequence valid for audience choice

Recall: **Claim:** There is a matching from LHS to RHS if each subset of LHS has neighbor set of larger cardinality.

- LHS: $\binom{5}{1}4! = 120$ edges from a LHS vertex
- RHS: 48 edges from a RHS vertex

Consider subset S of LHS.

- $120|S|$ edges leave subset
- $48|N(S)|$ enter neighbor set

so $|N(S)| > |S|$, and there is a matching by Hall's Theorem.

Real trick:

- At least 2 cards have same suit, one of which is at most 6 hops clockwise from the other in a cycle.
- Hide clockwise card; reveal other card first.
- Use remaining three cards to indicate number of hops, encoding $1, \ldots, 6$ using total order (e.g., low-medium-high or low-high-medium).