EECS 310: Discrete Math **Lecture 12**
Counting

Reading: MIT OpenCourseWare 6.042 Chapter 14-15

Pigeon-hole

Pigeon-hole principle: If n objects are placed into r boxes, then at least $\lceil n/r \rceil$ must be in the same box.

[[*puzzle about white/black socks in a drawer, how many must you select before you have a matching pair? holes – colors, pigeons – socks.*]]

Claim: Five cities on an alien planet, at least 4 are in same hemisphere.

Proof: Draw great circle through two of five points, by pigeonhole two of remaining three cities are in same hemisphere. □

Claim: Consider numbers $\{1, 2, \ldots, 2n\}$. Then in any set A of size $n+1$, at least two elts are relatively prime.

Proof:

- Pigeons – elts of A
- Holes – hole i is for pigeons between i and $i+1$

That is, at least two numbers are 1 apart and hence relatively prime. □

Claim: Consider numbers $\{1, 2, \ldots, 2n\}$. Then in any set A of size $n+1$, there are always two numbers such that one divides the other.

Proof: For each $a \in A$, write $a = 2^k m$, where m is odd and $1 \leq m \leq 2n-1$.

- Pigeons – elts of A
- Holes – odd numbers m

By pigeon-hole, must be two numbers a and a' with same hole m. That is, $a = 2^k m$ and $a' = 2^{k'} m$, so one is a multiple of the other. □

Probability

Monty Hall Problem:

- three doors, one car, two goats
- you pick a door
- host opens another door
- should you stay or switch?

Assumptions:

- prize is behind a random door
- you pick a random door
- host opens a door with a goat
- if host can pick door, picks a random door

Four Step Method

Step 1: Defining the sample space.

Def: *outcome* is any possible result of random choices.

Def: *sample space* is set of all outcomes.

[[*sometimes useful to construct outcomes using tree diagram.*]]

Example: Tree diagram for Monty Hall, doors A, B, C

- root decides door for prize
- level one decides door you pick
- level two decides door host picks
- label leaves with path from root

Leaves are sample space

$\{(A, A, B), (A, A, C), (A, B, C), (A, C, B), \ldots\}$

[[*sometimes host has only one choice, so only one leaf of some level one nodes*]]

Step 2: Defining events of interest.

Def: *event* is set of outcomes of interest

Example:

- prize is behind door A:
  $\{(A, A, B), (A, A, C),$
  $(A, B, C), (A, C, B)\}$
- you picked right door first time:
  $\{(A, A, B), (A, A, C), (B, B, A),$
  $(B, B, C), (C, C, A), (C, C, B)\}$
- you win by switching
  $\{(A, B, C), (A, C, B), (B, A, C),$
  $(B, C, A), (C, A, B), (C, B, A)\}$

[[*half outcomes in event that we win by switching, so might think prob is 1/2, but not all outcomes have same probability*]]

Step 3: Determine outcome probabilities

Modeling problem.

- assign edge probabilities
- compute outcome probabilities by product

[[*product because process progresses randomly along path*]]

Example: Compute outcome probabilities for Monty Hall

Step 4: Compute event probabilities

By summing outcomes in event.

Example: Compute $\Pr\{\text{switching wins}\}$ in Monty Hall

Probability Spaces

Def: countable *sample space* S is non-empty countable set, $w \in S$ is *outcome*

[[*countable means you can list the elements, also defined for uncountable sets but not in 310*]]

Def: *probability function* $\Pr\{\} : S \rightarrow [0, 1]$ is s.t.

- $\forall w \in S, \Pr\{w\} \geq 0$

- $\sum_{w \in S} \Pr\{w\} = 1$

Def: *probability space* is sample space plus probability function

Def: probability of event E

$$\Pr\{E\} = \sum_{w \in E} \Pr\{w\}$$

Implications

Claim: (Sum rule) if $E \cap F = \emptyset$, $\Pr\{E \cup F\} = \Pr\{E\} + \Pr\{F\}$

Claim: (Complement rule) $\Pr\{A\} = 1 - \Pr\{A^c\}$ where $A^c = \{w | w \notin A\}$

[[*can you come up with others? e.g., inclusion/exclusion?*]]

Claim: (Union bound)

$$\Pr\{E_1 \cup \ldots \cup E_n\} \leq \sum_{i=1}^{n} \Pr\{E_i\}$$

Uniform Probability Space

Def: probability space *uniform* if $\Pr\{w\}$ same for all $w \in S$

Claim: for uniform prob., $\Pr\{E\} = |E|/|S|$

Infinite Probability Space

Example: coin flipping

- two players alternate flipping coin until comes up heads
- person who flips heads wins

Draw tree diagram.

- infinite tree
- outcome space $T^n H$
- probability function?

$$\Pr\{T^n H\} = (1/2)^{n+1}$$

- valid? non-negative and

$$\sum_{i=0}^{\infty} (1/2)^{n+1} = (1/2) \sum_{i=0}^{\infty} (1/2)^n = 1$$

Question: prob. player 1 wins?

$$\begin{aligned} \Pr\{1 \text{ wins}\} &= \frac{1}{2} + \frac{1}{8} + \ldots \\ &= \frac{1}{2} \sum_{i=0}^{\infty} \left(\frac{1}{4}\right)^i \\ &= \frac{2}{3} \end{aligned}$$

Question: alternate solution?

- let p be prob. 1 wins, q be prob. 2 wins
- one of them wins: $p + q = 1$
- 1 flip, then 2 is first player: $p = (1/2) + (1/2)q$
- solve eqns, $p = 2/3$

Conditional Probability

Example: Prob. person p

- A =lives in Evanston?
- B =goes to Northwestern?
- B *given* A?

Draw Venn diagram.

Def: *prob. A given B,*

$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]}$$

Claim: (Product Rule)

$$\Pr[A \cap B] = \Pr[A|B]\Pr[B] = \Pr[B|A]\Pr[A]$$

Claim: (Product Rule)

$$\Pr[E_1 \cap \ldots \cap E_n] =$$

$$\Pr[E_1]\Pr[E_2|E_1]\ldots\Pr[E_n|E_1 \cap \ldots \cap E_{n-1}]$$

Note: Justifies tree diagram

- prob. on edge is conditional probability of edge given reach parent node
- multiple to get outcome prob. by above defn.

Example: A family has two children and one is a boy. What is probability other is a boy?

- $\Omega = \{BB, BG, GB, GG\}$
- $A = \{BB, BG, GB\}$
- $B = \{BB\}$
- $P(A) = 3/4$
- $P(A \cap B) = P(\{BB\}) = 1/4$
- $P(B|A) = 1/3$

Claim: Law of alternatives: If $A_1, \ldots, A_n$ are disjoint events whose union is S, then

$$P(B) = \sum_{i=1}^{n} P(A_i)P(B|A_i)$$

Claim: Bayes Thm: $P(A|B) = P(B|A)P(A)/P(B)$

Proof: $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$

Example: Cards: blue/blue, blue/red, red/red.

Question: Random side of random card is red, what is prob. other side is red?

- card 1: two blue, label sides front/back
- card 2: two red, front/back
- card 3: red side front, blue side back
- $S = \{(i, x)\}$ where i is card, x is front or back
- A = red side shown = $\{2F, 2B, 3F\}$, $P(A) = 1/2$
- B = card 2 chosen, = $\{2F, 2B\}$
- $P(B|A) = P(A \cap B)/P(A) = (1/3)/(1/2) = 2/3$