EECS 310: Discrete Math **Lecture 14**
Probability

Reading: MIT OpenCourseWare 6.042 Chapter 17-18

Random Variables

Def: *random variable* maps outcomes to values.

Note: It is a *function*!

Example: 3 indep. uniform coin tosses

- $S = \{(HHH), (HHT), \ldots, (TTT)\}$
- random variable: $M : S \to \{0, 1\}$,

  $$M(w) = \begin{cases} 1 & \text{if } w_1 = w_2 = w_3 \\ 0 & \text{otherwise} \end{cases}$$

  M indicates if coins all match.

- random variable: $C : S \to \{0, 1, 2, 3\}$,

  $$C(w) = \#H \in w.$$

  C counts heads.

R.V.'s define events

Def: *indicator random variable* maps outcomes to 0 or 1.

Note: for any event A, can associate indicator random variable I_A:

$$I_A(w) = \begin{cases} 1 & \text{if } w \in A \\ 0 & \text{otherwise} \end{cases}$$

Example: $M = I_E$ indicator random variable for event E that all coins match.

Note: more generally, $\{w : M(w) = k\}$ is an event.

Example: $C = 0$ is event "no heads"

Probability with R.V.'s

prob. of event defined by r.v.:

- $P[M = 1] = P[(TTT)] + P[(HHH)] = 1/4$
- $P[C = 2] = 3/8$

Note: conditional probability, independence apply to random variables too.

Def: Random variables X and Y are independent iff *for all* x and y, $P[X = x \cap Y = y] = P[X = x]P[Y = y]$ iff *for all* x and y, $P[X = x|Y = y] = P[X = x]$.

Example: Are M and C independent?

Note: Must find m and c such that the inequality fails:

$$P[C = 2 \cap M = 1] = 0, P[C = 2]P[M = 1] > 0$$

Functions of R.V.'s

Note: Functions of r.v.'s are r.v.'s.

Example: 2 indep. unbiased 6-sided dice

Some r.v.'s:

- D_i = number on i'th die
- $T = D_1 + D_2$ total of 2 dice
- e^T, other crazy (but useful) r.v.

Balls and Bins Example

Hashing again:

- n keys (bins), n records (balls)
- random hash function (ball goes to random bin)
- resolve collisions with linked list (want to bound max bin size)

Question: how long are the linked lists?

- let X be rand. var. equal to max size.
- find smallest k s.t. $P[X \geq k] \leq 1/2$
- Let X_i = size of bin i.
- Then
$$(X \geq k) =$$
$$(X_i \geq k) \cup \ldots \cup (X_n \geq k).$$
- By *union bound*
$$P[X \geq k] \leq \sum_{i=1}^{n} P[X_i \geq k]$$

so find k s.t. $P[X_i \geq k] \leq \frac{1}{2n}$.

$$\begin{aligned} P[X_i \geq k] &= P[(X_i = k) \cup (X_i = k+1) \cup \\ &\quad \ldots \cup (X_i = n)] \\ &= \sum_{j=k}^{n} P[X_i = j] \\ &= \sum_{j=k}^{n} \binom{n}{j} \left(\frac{1}{n}\right)^j \left(1 - \frac{1}{n}\right)^{n-j} \\ &\leq \sum_{j=k}^{n} \left(\frac{1}{j}\right)^j \\ &\leq \sum_{j=k}^{n} \left(\frac{1}{k}\right)^j \\ &= \left(\frac{1}{k}\right)^k \sum_{j=0}^{\infty} \left(\frac{1}{k}\right)^j \\ &\leq 2\left(\frac{1}{k}\right)^k \end{aligned}$$

Conclusion:
$$2\left(\frac{1}{k}\right)^k \leq \frac{1}{2n}$$
if
$$k \sim \ln n / \ln \ln n$$

E.g., $n = 10^6$, w.p.r. 1/2, just 10 collisions max.

Expectation

Def: *expected value* of r.v. R is

$$\begin{aligned} E[R] &= \sum_{w \in S} R(w) P[w] \\ &= \sum_{k=1}^{\infty} P[R \geq k] \end{aligned}$$

(last holds for non-neg only).

Example: 6-sided die, R value of roll

$$E[R] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \ldots + 6\frac{1}{6} = \frac{7}{2}$$

[[*R may never take value $E[R]$.*]]

Example: indicator r.v. I_A for event A

$$E[I_A] = 1 \cdot \Pr[I_A = 1] + 0 \cdot \Pr[I_A = 0]$$
$$= \Pr[I_A = 1] = \Pr[A]$$

Claim: If $R : S \to \text{range}(R)$, then

$$E[R] = \sum_{x \in \text{range}(R)} x \cdot \Pr[R = x]$$

Proof:

$$\begin{aligned} E[R] &= \sum_{w \in S} R(w) \Pr[w] \\ &= \sum_{x \in \text{range}(R)} \sum_{w \in [R=x]} R(w) \Pr[w] \\ &= \sum_{x \in \text{range}(R)} \sum_{w \in [R=x]} x \Pr[w] \\ &= \sum_{x \in \text{range}(R)} x \left(\sum_{w \in [R=x]} \Pr[w] \right) \\ &= \sum_{x \in \text{range}(R)} x \Pr[R = x] \end{aligned}$$

Example: R is # heads in n flips of coin with bias p (binomial dist.)

$$E[R] = \sum_{k=0}^{n} k \Pr[R = k]$$

$$= \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

[[*ugly, see another way later*]]

Claim: If $\text{range}(R) = \mathbb{N}$, then

$$E[R] = \sum_{i=1}^{\infty} i \Pr[R = i] = \sum_{i=0}^{\infty} \Pr[R > i]$$

Proof: sum equations:

$$\Pr[R > 0] = \Pr[R = 1] + \Pr[R = 2] + \ldots$$

$$\Pr[R > 1] = \Pr[R = 2] + \Pr[R = 3] + \ldots$$

$$\Pr[R > 2] = \Pr[R = 3] + \Pr[R = 4] + \ldots$$

Example: Mean time to failure (geometric dist)

Question: Planes crash with prob. p, what's the number of flights F until you die?

$$E[F] = \sum_{i=0}^{\infty} \Pr[F > i] = \sum_{i=0}^{\infty} (1-p)^i = \frac{1}{p}.$$

[[*plane crashes after i'th iff didn't crash during first i flights*]]

Claim: Linearity of expectation: For any two r.v. R_1 and R_2, $E[R_1 + R_2] = E[R_1] + E[R_2]$.

Proof: Let $T = R_1 + R_2$.

$$\begin{aligned} E[T] &= \sum_{w \in S} T[w] P[w] \\ &= \sum_{w \in S} (R_1[w] + R_2[w]) P[w] \\ &= \sum_{w \in S} R_1[w] P[w] + \sum_{w \in S} R_2[w] P[w] \\ &= E[R_1] + E[R_2] \end{aligned}$$

Claim: $E[aX] = aE[X]$

Fact: Expectation is a linear function.

[[*No independence required!*]]

Example: R is # heads in n flips of coin with bias p (binomial dist.)

- R_i is flip of i'th coin
- $R = \sum_{i=1}^{n} R_i$
- $E[R] = E[\sum_i R_i] = \sum_i E[R_i] = np$

[[*nicer form, and proof that ugly sum from before equals np*]]

Example: Dim sum: n people at a round table, each has a teacup. Table is spun. X is number of people who get their own teacup back.

- X_i = indicator r.v. that i'th person gets own teacup

- X_i binomial with prob. $1/n$
- $X = \sum_i X_i$, and $E[X] = n(1/n) = 1$

Note: X_i *not* independent, still $E[X] = \sum_i E[X_i]$

Example: Coupon collector:

n types of coupons, X = number must collect to get one of each.

- X_i = number of steps between getting i'th coupon and $(i+1)$'st coupon

$$R, R, G, B, G, Y \rightarrow$$
$$X_0 = 1, X_1 = 2, X_2 = 1, X_3 = 2$$

- $X = \sum_{i=0}^{n-1} X_i$
- When looking for $(i+1)$'st new coupon, already have i types, so get new type each draw with probability $1 - \frac{i}{n}$.
- $E[X_i] = \frac{n}{n-i}$

  (geometric dist. with $p = \frac{n-i}{n}$)
- $E[X] = \sum_{i=0}^{n-1} E[X_i]$
  $= \sum_{i=0}^{n-1} \frac{n}{n-i} = n \sum_{j=1}^{n} \frac{1}{j} = O(n \ln n)$