EECS 310: Discrete Math **Lecture 2**
Logic, Proofs

Reading: MIT OpenCourseWare 6.042 Chapter 2

Logic Recap

Let P and Q be propositions.

Logical operators:

- Not: $\neg P$
- And: $P \wedge Q$
- Or: $P \vee Q$
- Implication/Conditional: $P \rightarrow Q$ (if P then Q)
- Equivalence/Biconditional: $P \leftrightarrow Q$ (P if and only if Q)

Other terminology: Given implication $P \rightarrow Q$,

- Converse is $Q \rightarrow P$
- Contrapositive is $\neg P \rightarrow \neg Q$

[[*Which is logically equivalent to implication?*]]

Let D be domain of discourse, $P(x), Q(y), R(x, y)$ be predicates.

Quantifiers:

- Existential: $\exists$
- Universal: $\forall$

Mixing quantifiers:

- $\forall x \in D, \exists y \in D, R(x, y)$
- $\exists y \in D, \forall x \in D, R(x, y)$

Negation:

- $\neg(\exists x \in D, P(x)) = \forall x \in D, \neg P(x)$
- $\neg(\forall y \in D, Q(y)) = \exists y \in D, \neg Q(y))$
- $\neg(\forall x \in D, \exists y \in D : R(x, y)) = \exists x \in D, \forall y \in D : \neg R(x, y)$

Proofs

Proof recipes:

1. Direct proof: To prove $P \rightarrow Q$, assume P and derive Q.
2. Proof by contrapositive: To prove $P \rightarrow Q$, prove $\neg Q \rightarrow \neg P$ by direct proof.
3. Proof by contradiction: To prove P, assume $\neg P$ and derive a contradiction.
4. Proof by cases: Break into substatements and prove each individually using one of above recipes.

1. Direct proof: To prove $P \to Q$, assume P and derive Q.

   Example: Claim: If n is odd, then $n+1$ is even.

   Proof:

   - By direct proof.
   - Assume n is odd.
   - Then $n = 2k+1$ for some integer $k \Rightarrow n+1 = 2k+2 = 2(k+1) \Rightarrow n+1$ is even.

   $\square$

2. Proof by contrapositive: To prove $\neg Q \to \neg P$:, assume $\neg Q$ and derive $\neg P$.

   Example: Claim: Let d divide n. If n is odd, then d is odd.

   Proof:

   - Proof by contrapositive: if d is even, then n is even.
   - Assume d is even and d divides n.
   - Then $d = 2k$ for some integer k (d is even) and $dl = n$ for some integer l (d divides n) $\Rightarrow n = l(2k) = 2(kl)$ $\Rightarrow n$ is even.

   $\square$

3. Proof by contradiction: To prove P, assume $\neg P$ and reach a logical contradiction.

   Example: Claim: Alice wins in Chomp.

   Proof:

   - Proof by contradiction.
   - Assume Alice loses, i.e. for any first move of Alice, Bob can force a win.
   - Then consider Bob's response (i, j) when Alice chooses (m, n) as her first move.
   - By assumption, Bob forces a win from (i, j), i.e., he has a sequence of responses for any response of Alice that guarantees he wins.
   - But then Alice could have chosen (i, j) as her first move and copied Bob's strategy.
   - Contradiction: By copying Bob, Alice wins.

   $\square$

   Example: Claim: There is no greatest even integer.

   Proof:

   - Proof by contradiction.
   - Assume there is a greatest even integer N.
   - Then $N = 2k$ for some integer $k \Rightarrow N+2 = 2k+2 = 2(k+1)$ is even and greater than N.
   - Contradiction: $N+2$ is an even integer greater than N.

   $\square$

4. Proof by cases: Break into logical substatements and prove each individually.

   Example: Claim: For any two numbers x and y, $(x+y)/2 \leq \max(x, y)$.

   Proof:

   - By cases.
   - First rewrite as $(x+y) \leq 2\max(x, y)$.
   - Case 1: $x \leq y$. Then $(x+y) \leq (y+y) = 2\max(x, y)$.

- Case 2: $x > y$. Then $(x + y) < (x + x) = 2\max(x, y)$.

□

[[*Also proof by contradiction.*]]

More Proofs

Example:

Claim: If the sum of the digits of a number n is divisible by 9, then n is divisible by 9.

Proof:

- By direct proof.
- Assume the sum of the digits of n are divisible by 9.
- Then $x_0 + \ldots + x_l = 9k$ for some integer k where $x_0, \ldots, x_l$ are the digits of $n \Rightarrow n = x_0 + 10x_1 + \ldots + 10^l x_l = (x_0 + \ldots + x_l) + 9x_1 + \ldots + 99...9x_l = 9k + 9(x_1 + \ldots + 11...1x_l) = 9(k + x_1 + \ldots + 11...1x_l)$.

□

[[*A proof from the book – Erdos said there was The Book in which God maintains the perfect proofs of mathematical theorems. He has a famous quote that "you need not believe in God but, as a mathematician, you should believe in The Book".*]]

Example:

Claim: There are infinitely many primes.

Proof:

- By contradiction.
- Assume there are finitely many primes $p_1, \ldots, p_n$.
- Consider $q = p_1 p_2 \ldots p_n + 1$. Then q is larger than all p_i, so not a prime. Hence q has a prime divisor $p = p_i$ for some i. Since $p|q$ and $p|p_1 p_2 \ldots p_n$, $p|1$ which is a contradiction.

□

Example:

Claim: If p^2 is even, then p is even.

[[*Exercise: prove above claim.*]]

Claim: $\sqrt{2}$ is irrational.

Proof:

- Proof by contradiction.
- Assume $\sqrt{2}$ is rational.
- Then $\sqrt{2} = p/q$ for two integers p and q. Choose p and q so that they are not both even. Then $p^2 = 2q^2$ so p^2 is even.
- p^2 is even $\Rightarrow p$ is even $\Rightarrow p = 2k$ for some integer $k \Rightarrow p^2 = 4k^2 \Rightarrow 2q^2 = 4k^2 \Rightarrow q^2 = 2k^2 \Rightarrow q$ is even.
- Contradicts the assumption that not both p and q are even.

□

Example:

Claim: There exist irrational numbers x and y such that x^y is rational.

Proof:

- Proof by cases.
- Let $x = \sqrt{2}$ and $y = \sqrt{2}$.

  1. $\sqrt{2}^{\sqrt{2}}$ is rational. Then x and y are irrational and yet x^y is rational.

2. $\sqrt{2}^{\sqrt{2}}$ is irrational. Then let $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$. Then they are both irrational and yet $x^y = \sqrt{2}^2 = 2$ is rational.

$\square$

[[*Notice we showed existence without concluding which pair is irrational!*]]