EECS 310: Discrete Math **Lecture 3**
Proofs, Sets, Functions, Induction

Reading: MIT OpenCourseWare 6.042 Chapter 3.1-3.2

Proof Review

Recipes:

1. Direct proof: To prove $P \to Q$, assume P and derive Q.
2. Proof by contrapositive: To prove $P \to Q$, prove $\neg Q \to \neg P$ by direct proof.
3. Proof by contradiction: To prove P, assume $\neg P$ and derive a contradiction.
4. Proof by cases: Break into substatements and prove each individually using one of above recipes.

[[*Miniquiz time. Go over below solution after miniquiz (have someone volunteer to write it on the board).*]]

Sets

[[*Set theory formalized by Georg Cantor in 1895 who died in a mental hospital in 1918.*]]

Def: A *set* is a collection of elements.

Example:

- Letters of the alphabet $\{a, b, \ldots, z\}$
- Natural numbers $\mathbb{N} = \{0, 1, \ldots\}$
- Integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- Reals $\mathbb{R}$
- Rationals $\mathbb{Q} = \{p/q : p, q \in Z\}$

  Note notation.

Note:

- no order: $\{a, b\} = \{b, a\}$
- no repetition: $\{a, a\} = \{a\}$

Membership:

- x is an element of set A: $x \in A$
- examples: $3 \in \mathbb{N}$, $\sqrt{2} \notin \mathbb{Q}$

Set builder notation:

Define set using predicate – set is everything s.t. predicate is true:

Even $\#'s = \{n \in \mathbb{N} : n = 2k \text{ for some } k \in \mathbb{N}\}$

Odd $\#'s = \{n \in \mathbb{N} : n = 2k+1 \text{ for some } k \in \mathbb{N}\}$

Comparing and Combining

Set operations:

- union: $A \cup B = \{x : x \in A \vee x \in B\}$

- intersection: $A \cap B = \{x : x \in A \wedge x \in B\}$
- complement: $A^c = \{x : x \notin A\}$

Venn diagrams:

- $A = \{1, 2, 3\}$
- $B = \{3, 4, 5\}$

Disjoint: intersection is empty.

- even and odd integers disjoint

Subset: $A \subseteq B$ if and only if every element of A is an element of B.

$$[A \subseteq B] \leftrightarrow [\forall x : x \in A \rightarrow x \in B]$$

[[*like $\leq$ sign*]]

Strict subset: $A \subset B$ if $A \subseteq B$ and $\exists x : x \in B$ and $x \notin A$.

[[*like $<$ sign*]]

Equality: $\forall x, x \in A \iff x \in B$.

- even $\cup$ odd $= \mathbb{N}$

[[*prove equality by proving biconditional, each subset of other.*]]

Representation

Computer representation:

- Universe $U = \{1, 2, 3\}$ as master set
- Subset as vector of zero-one, e.g., $\{1, 2\}$ is vector 110.

Containment? Union? Intersection?

Well-Ordering Principle

Claim: Every *nonempty* set of *nonnegative* integers has a smallest element.

[[*Obvious? Not true for every set, e.g., negative integers, rational numbers.*]]

Proof by least counterexample

Example:

Claim: For any integers m and n, the fraction m/n can be written in lowest common form.

Check: $m = 10, n = 20$, fraction $10/20 = 1/2$.

Proof: By least counter-example.

- Let $C = \{m | \exists n : m/n$ has no lowest common form$\}$.
- Assume by way of contradiction that $C \neq \emptyset$.
- By well-ordering, C has smallest element m_0.
- Let n_0 be such m_0/n_0 can not be written in lowest common form.
- Then $\exists p$ such that $p | m_0$ and $p | n_0$.
- Then $m_0/n_0 = (m_0/p)/(n_0/p)$ so this also can't be written in lowest common form.
- Thus $m_0/p \in C$.
- But $m_0/p < m_0$, contradicting that m_0 was smallest element in C.

Template: To prove $P(n)$ by least counter-example,

1. Define set $C = \{n | \neg P(n)\}$ of counterexamples to $P(n)$.

2. Use proof by contradiction to assume $C \neq \emptyset$.

3. Use well-ordering to claim $\exists$ smallest element $n \in C$.

4. Reach contradiction (usually by using n to construct a smaller element in C).

5. Conclude $C = \emptyset$.

Induction

[[*Line of dominoes: first falls, if n'th falls then $(n+1)$'st falls, so all fall.*]]

Notation:

- Sum: for numbers $x_1, \ldots, x_n$:

$$x_1 + \ldots + x_n = \sum_{i=1}^{n} x_i.$$

- Product: for numbers $x_1, \ldots, x_n$:

$$x_1 \cdot \ldots \cdot x_n = \prod_{i=1}^{n} x_i.$$

Example: Let $P(n)$ be the predicate that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Claim: $\forall n \in Z^+, \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Proof:

- For $n = 1$, $\sum_{i=1}^{1} i = 1 = \frac{1(2)}{2}$.

- For $n = 2$, $1 + 2 = \frac{1(2)}{2} + 2 = \frac{1(2)}{2} + \frac{4}{2} = \frac{1(2)+2(2)}{2} = \frac{2(3)}{2}$.

- For $n = 3$, $1+2+3 = \frac{2(3)}{2} + 3 = \frac{2(3)+2(3)}{2} = \frac{3(4)}{2}$.

- For $n + 1$,
  - Assume $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.
  - Then:

$$\begin{aligned} \sum_{i=1}^{n+1} i &= \sum_{i=1}^{n} i + n + 1 \\ &= \frac{n(n+1)}{2} + n + 1 \\ &= (n+1)\left(\frac{n}{2} + 1\right) \\ &= \frac{(n+1)(n+2)}{2}. \end{aligned}$$

[[*Mention Gauss's proof, i'th number plus $(n-i+1)$'th number sum to $n+1$ and there are $n/2$ pairs.*]]

Template: To prove $P(n)$ by induction,

- Prove base case $P(1)$.

- State inuductive hypothesis $P(n)$.

- Prove $P(n) \to P(n+1)$ for all $n \geq 1$.

- Conclude $P(n)$ for all $n \geq 1$.

Example: Let $P(n)$ be the predicate that $\sum_{i=1}^{n}(2i-1) = k^2$ for some integer k.

Claim: Sum of first n odd integers is a perfect square.

Proof:

- By induction.

- Base case $P(1)$: $\sum_{i=1}^{1}(2(1)-1) = 1 = 1^2$ is a perfect square.

- Inductive hypothesis $P(n)$: $\sum_{i=i}^{n}(2i - 1) = k^2$ for some integer k.

- Inductive step $P(n) \to P(n+1)$: $\sum_{i=1}^{n+1}(2i-1) = \sum_{i=1}^{n}(2i-1) + 2(n+1) - 1 = k^2 + 2n - 1 = \ldots?$

□

Hard part: what to prove?

[[*Mention geometric intuition, derive that sum is n^2.*]]

Take two:

Proof: We prove the stronger predicate $P'(n)$ that $\sum_{i=1}^{n}(2i-1) = n^2$.

- By induction.
- Base case $P(1)$: $\sum_{i=1}^{1}(2(1)-1) = 1 = 1^2$.
- Inductive hypothesis $P(n)$: $\sum_{i=i}^{n}(2i-1) = n^2$.
- Inductive step $P(n) \rightarrow P(n+1)$: $\sum_{i=1}^{n+1}(2i-1) = \sum_{i=1}^{n}(2i-1) + 2(n+1) - 1 = n^2 + 2n - 1 = (n+1)(n+1)$.

□

Example: Tiling with L-shapes.

Claim: Any $2^n \times 2^n$ foot bathroom can be tiled with L-shapes, leaving a center square empty for a rubber duck statue.

[[*Draw 2×2 and 4×4 examples.*]]

[[*Hard to prove because chop up into four sub-squares, doesn't leave center tile empty.*]]

Claim: Any $2^n \times 2^n$ bathroom can be tiled leaving *any* square empty.

Proof: By induction. Let $P(n)$ be the predicate in claim.

- Base case $P(1)$: see example.
- Inductive hypothesis: True for $2^n \times 2^n$ bathrooms.
- Inductive step:
  - Break $2^(n+1) \times 2^(n+1)$ bathroom up into four 2^n subsquares.
  - Desired empty square is in one, tile it using inductive hypothesis.
  - Tile other three leaving corners empty using inductive hypothesis.
  - Fill corners using an L-shaped tile.

Hard part: what to do induction on?

Example: Two-color theorem.

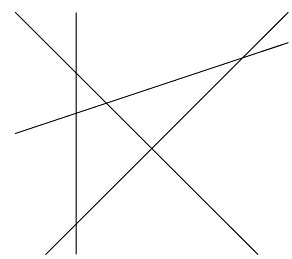

Figure 1: Map to be colored

Claim: Any map drawn with straight lines can be colored with just 2 colors.

Proof:

- By induction on number of lines: $P(n) =$ map with n lines can be colored.
- Base case $P(0)$: Map with 0 lines has 1 region, color it blue.
- Inductive hypothesis $P(n)$: Map with n lines can be colored.
- Inductive step $P(n) \rightarrow P(n+1)$: Given map with $(n+1)$, remove a line. Color by IH. Replace line and flip all colors to one side.
  1. If a region was not split by line, then still ok.
  2. If a region was split by line, two halves have different colors, and bordering regions still ok.

□

[[*Discuss four-color theorem.*]]

Technique:

[[*Be careful not to assume what you want to prove!*]]

Claim: $\forall x \geq 0, n \in \mathbb{N}, (1 + nx) \leq (1 + x)^n$

Pitfalls:

Claim: All horses are the same color.

Proof:

- Let $P(n)$ = all sets of n horses have same color.
- $P(1)$ is true since only one horse.
- $P(n + 1)$:
  - number horses from 1 to $n + 1$
  - sets $S_1 = \{1, \ldots, n\}$ and $S_2 = \{2, \ldots, n + 1\}$ have n horses, same color by IH
  - sets overlap so color of S_1 must equal color of S_2

□

[[*Inductive step not complete (for $n = 2$, sets don't overlap).*]]