EECS 310: Discrete Math	**Lecture 4**
Sets, Induction	

Reading: MIT OpenCourseWare 6.042 Chapter 3.3-3.5

Sets Review

Universe $U = \{0, 1, 2, 3\}$ with bit vector $x = x_0x_1x_2x_3$.

Sets $A = \{1, 2\}, x_A = 0110$, and $B = \{2, 3\}, x_B = 0011$.

Set concepts:

- union $A \cup B = \{1, 2, 3\}, x_{A\cup B} = 0111$
- intersection $A \cap B = \{2\}, x_{A\cap B} = 0010$
- complement $A^c = \{0, 3\}, x_{A^c} = 1001$

Induction Review

Basic Induction:

Want to prove $P(n)$.

- Prove base case $P(1)$.
- Prove $P(n) \to P(n+1)$ (by direct proof).
  - Inductive hypothesis: assume $P(n)$.
  - Inductive step: using hypothesis, derive $P(n + 1)$.

Invariants by Induction

[[*Useful to prove algorithm is correct.*]]

Example: Robot moves on diagonals of grid, starting at $(0, 0)$.

Claim: Robot never steps on flower at $(0, 1)$.

States after

- 1 move: $(1, 1), (1, -1), (-1, 1), (1, 1)$
- 2 moves: $(0, 0), (0, 2), (2, 2), (2, 0), \ldots$
- etc.

Sum of coordinates always even!

Predicate $P(t)$: After t steps, if robot is at (x, y), then $x + y$ is even.

Claim: Sum of coordinates always even.

Proof: By induction.

- Base case: $P(0)$ is true since starting position $(0, 0)$ is $0 + 0 = 0$ is even.
- Inductive hypothesis: after t steps, robot is at (x, y) where $x + y$ is even.
- Inductive step: by cases.
  - Robot moved northwest. New position is $(x - 1, y + 1)$. Sum is $x + y$, even by hypothesis.
  - Robot moved northeast. New position is $(x+1, y+1)$. Sum is $x+y+2$, even.

– etc.

Since $1 + 0 = 1$ is odd, robot never steps on flower.

Example: The 8-puzzle: slide tiles to convert

A	B	C
D	E	F
H	G	.

into

A	B	C
D	E	F
G	H	.

Claim: Not possible.

Note: Row moves don't change order. **Note:** Column moves change order of two pairs.

Def: Tiles T_1 and T_2 are inverted if out-of-alphabetical order.

A	B	C
F	D	G
E	H	.

Has three inversions: (D, F), (E, F), (E, G).

Claim: Moves change number of inversions by 2 or 0.

Proof:

- Row move doesn't change number.
- Column moves switch exactly two pairs:
  - If both pairs originally inverted, total number of inversions decreases by 2.
  - If just one pair originally inverted, it gets sorted and other gets inverted, total doesn't change.
  - etc.

Claim: In every configuration reachable by legal moves, parity of number of inversions is odd (i.e., sum is an odd number).

Proof: By induction.

- Base case: initial configuration has 1 inversion.
- Inductive hypothesis: after t moves, odd parity.
- Inductive step: by above claim, number changes by 2 or 0, so $t + 1$'th move has odd parity by inductive hypothesis.

Sorted board not reachable since parity is even.

Strong Induction

Useful when predicate $P(n+1)$ naturally depends on some $m < n$.

Suppose you want to prove $P(n)$.

- Prove base case $P(1)$.
- Inductive hypothesis: assume $P(m)$ for all $1 \le m \le n$.
- Inductive step: using hypothesis, derive $P(n+1)$.

Example: Prime factorization.

Claim: Every integer $n > 1$ is product of primes.

Proof: By strong induction.

- Base case $P(2)$: $2 = 1 \times 2$ is product of primes.

- Inductive hypothesis: m is product of primes for all $2 \leq m \leq n$.
- Inductive step:
  - If $n+1$ prime, done.
  - If not, then $n+1 = km$ for some integers $k, m \in \{2, 3, \ldots, n\}$.
  - By inductive hypothesis, k, m are products of primes, and thus so is $n+1$.

Example: Making change.

Claim: Every amount of postage of 12 cents or more can be formed using just 4 and 5 cent stamps.

Proof: By strong induction

- $P(n) = n$ cents of postage formed with $4, 5$ cent stamps
- $P(n)$ true for $n \in \{12, 13, 14, 15\}$
- assume $P(k)$ for all $k \leq n$
- $P(n+1)$: use IH to get $n-3$ cents of postage and add a 4 cent stamp

□

Claim: It takes at most $nm - 1$ breaks to divide an n-by-m chocolate bar.

Proof:

- By strong induction on number k of squares in bar.
- Base case: With 1 square, need $1 \cdot 1 - 1 = 0$ breaks.
- Inductive hypothesis: Assume any bar with at most k squares can be divided with $k-1$ breaks.
- Inductive step:
  - Given a bar with $k+1$ squares, use one break to get two bars with s_1 and s_2 squares respectively where $s_1 + s_2 = k + 1$.
  - Use inductive hypothesis to break these with $s_1 - 1$ and $s_2 - 1$ breaks respectively.

  So used $1 + (s_1 - 1) + (s_2 - 1) = s_1 + s_2 - 1 = (k+1) - 1$ breaks.

□

Structural Induction

Induction on recursively-defined data types.

Example: parantheses.

Def: Set M of matched parenthetical statements:

- empty string λ is in M
- if $s, t \in M$, then $(s)t \in M$

So

- $() \in M$ using $s = t = \lambda$
- $()() \in M$ using $s = \lambda, t = ()$
- $(()) \in M$ using $s = (), t = \lambda$
- etc.

Template:

- Prove for base cases of definition.
- Prove for constructor case assuming holds for component types.

Claim: $\forall s \in M, s$ has equal number of open and close parantheses.

Proof: By induction.

- Base case: λ has zero open and zero close paranetheses.
- Constructor case: must show $P(r)$ for $r = (s)t$ assuming $P(s)$ and $P(t)$.
  - Let n_s, n_t be of open parantheses (= number close parantheses by hypothesis) in s, t respectively.
  - Then number of open paranetheses in expression is $n_s + n_t + 1$.
  - Similarly for close parantheses.