EECS 310: Discrete Math **Lecture 5**
Graph Theory, Matching

Reading: MIT OpenCourseWare 6.042 Chapter 5.1-5.2

Induction Review

Basic Induction:

Want to prove $P(n)$.

- Prove base case $P(1)$.
- Prove $P(n) \to P(n+1)$ (by direct proof).
  - Inductive hypothesis: assume $P(n)$.
  - Inductive step: using hypothesis, derive $P(n+1)$.

Strong Induction:

Want to prove $P(n)$.

- Prove base case $P(1)$.
- Inductive hypothesis: assume $P(m)$ for all $m \leq n$.
- Inductive step: using hypothesis, derive $P(n+1)$.

Graphs

Def: A *simple graph* consists of

- a set of *vertices* or *nodes* V
- a set of *edges* or collection of two-elt subsets of vertices E.

Denote graph by $G(V, E)$.

Example: Draw a directed and undirected graph.

Note: By definition, undirected, no self-loops, no multi-edges

Terminology:

- *adjacent* nodes – share an edge
- edge *incident* to nodes it joins
- *degree* of node is number of incident edges
- G_1 *subgraph* of G_2 if $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$

Example: Find examples of each of these definitions in above graph.

Common Graphs

- Empty graph: no edges
- Complete graph: an edge between every pair of vertices (how many?)
- Line: n-node graph with $n-1$ edges:
  $V = \{v_1, \ldots, v_n\}$, $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}\}$
- Cycle: n-node graph with n edges:
  add edge $\{v_n, v_1\}$ to above

Isomorphism

Example: squares with different node labels, pentagon vs star.

Def: $G_1(V_1, E_1)$ is *isomorphic* to $G_2(V_2, E_2)$ if there's a function f mapping each $v \in V_1$ to a unique $v \in V_2$ (bijection) s.t.

$\forall u, v \in V_1, \{u, v\} \in E_1 \leftrightarrow \{f(u), f(v)\} \in E_2$.

Same up to relabeling.

[[*Isomorphism for examples? Simple checks? (number vertices same, degree sequence same) No easy-to-check necessary and suff condition.*]]

Matching Problems

Nodes – people Edges – u likes v Matching – marriages (no polygamy)

[[*Match packets to paths, traffic to web servers, etc.*]]

Sex in America

Question: Who has more (heterosexual) sex?

- UofC survey, 1994: men have 74% more partners
- ABC News, 2004: men have 20, women have 6
- NYTimes, 2007: men have 7, women have 4

Graph $G(V, E)$,

- V are people, split into M (men) and W (women), suppose no one is both or neither man or woman,
- $\{u, v\} \in E$ if u and v have sex.

Example: draw potential graph

Claim: Surveys are inaccurate.

Proof: Two ways to count edges:

- sum degrees of men: $\sum_{v \in M} deg(v)$
- sum degrees of women: $\sum_{v \in W} deg(w)$

Divide by $|M||W|$:

$$\frac{1}{|W|}\sum_{v \in M} deg(v)/|M| = \frac{1}{|M|}\sum_{v \in W} deg(v)/|W|$$
$$avedeg(v \in M) = \frac{|W|}{|M|} avedeg(v \in W)$$

So men have $\frac{|W|}{|M|} = 3.5\%$ more sex.

[[*Promiscuity irrelevant.*]]

Question: Minority students study more with non-minority students (Boston Globe). Surprising or not?

[[*Key observation for above, relationship between degrees and edges. More general:*]]

Claim: Handshaking Lemma: The sum of the degrees of vertices in a graph is twice the number of edges.

Proof: Each edge contributes twice to the sum of degrees, once for each end. □

Bipartite Graphs

Def: A graph is *bipartite* if there're subsets L and R of vertices s.t.

- $L \cap R = \emptyset$ and $L \cup R = V$
- every edge incident to one $u \in L$ and one $v \in R$.

Example: Boys and girls and relationships in a heterosexual world.

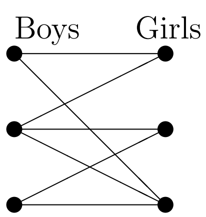
Question: Given

- n boys B
- n girls G
- Pairs E where $(b, g) \in E$ if boy b and girl g like each other.

Find

- a way to marry everyone off such that each couple likes each other.

Example:



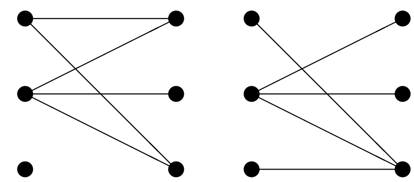
When can we do this?

- Not if some boy likes no girls
- Not if some two boys only like the same girls
- ... not if some k boys collectively like strictly less than k girls.

Example:

Last condition both necessary and sufficient!

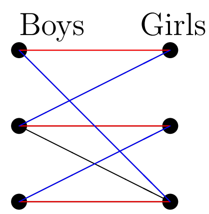
Def: A *bipartite graph* G has vertex set $V = V_1 \cup V_2$ and edge set E such that $\forall (u, v) \in E, u \in V_1, v \in V_2$ (all edges are between V_1 and V_2).



Def: $M \subseteq E$ is a *matching* iff each $v \in V_1 \cup V_2$ has at most one edge.

Def: $M \subseteq E$ is a *perfect matching* iff each $v \in V_1 \cup V_2$ has exactly one edge.

Example:



Notation: For a set of vertices S, let $N(S)$ be the *neighbors* of vertices in S.

$$N(S) = \{v : \exists u \in S \wedge (u, v) \in E\}.$$

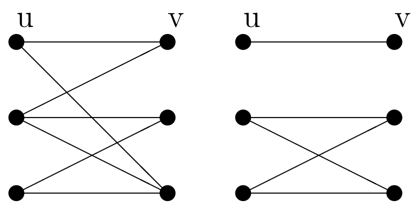
Claim: Hall's Theorem: For $G = (V_1, V_2, E)$ with $|V_1| = |V_2|$, there is a perfect matching in G iff $\forall S \subseteq V_1, |S| \leq |N(S)|$.

Proof:

- $\rightarrow$: For any set S, $N(S)$ includes all matched vertices, so $|S| \leq |N(S)|$.
- $\leftarrow$: By induction on $n = |V_1| = |V_2|$. Base case $n = 1$ holds. Assume true for graphs of size at most n. Two cases:
    1. $\forall S \subset V_1, |N(S)| \geq |S| + 1$: Pick $u \in V_1, v \in N(\{u\})$. Hall's condition holds for $G - \{u, v\}$, so use I.H.

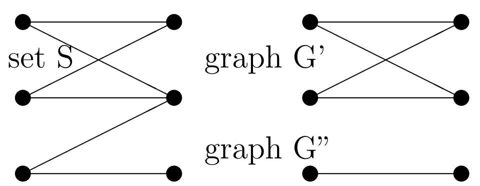
to find matching. Match u to v to get matching for G.

Example:



2. $\exists S \subset V_1, |S| = |N(S)|$: Split graph $G' = (S, N(S), E)$ and $G'' = (V_1 - S, V_2 - N(S), E)$.

   Example:



   - Hall's condition holds for G' so use I.H. to find matching M'.
   - Consider $T \subseteq V_1 - S$:

$$\begin{aligned} N''(T) &= N(T \cup S) - N(S) \\ &\geq |T \cup S| - |S| \\ &= |T| \end{aligned}$$

   so Hall's condition holds for G'' so use I.H. to find matching M''.

   Return matching $M' \cup M''$ for G.

$\square$

Claim: Every k-regular bipartite graph on vertices (V_1, V_2) has a perfect matching.

Proof: Each vertex has degree k, so

$$k|V_1| = |E| = k|V_2|.$$

As $k > 0$ this means $|V_1| = |V_2|$.

Let $S \subseteq |V_1|$, E_1 be edges adjacent to S and E_2 edges adjacent to $N(S)$. Then by definition of $N(S)$, $E_1 \subseteq E_2$, so

$$k|S| = |E_1| \leq |E_2| = k|N(S)|.$$

Hence $|S| \leq |N(S)|$ so graph has a perfect matching by Hall's theorem. $\square$

[[*Can skip below.*]]

Applications:

Def: A Latin rectangle is a $r \times s$ array of digits from $1 \ldots n$ such that any given digit appears at most once in every row and column.

Question: Given an $r \times n$ Latin rectangle, can it be extended to an $n \times n$ Latin square?

Example:

Rectangle:

1	3	2	4
2	1	4	3

Extension:

1	3	2	4
2	1	4	3
3	4	1	2
4	2	3	1

Claim: Every $r \times n$ Latin rectangle can be extended to a $n \times n$ Latin square.

Proof: Show can extend $r \times n$ to $(r+1) \times n$.

- Make a bipartite graph:
  - one node set C is columns of Latin rectangle
  - other node set D is digits from 1 to n
  - edge (c, d) for $c \in C, d \in D$ iff column c does not contain digit d

[[*Draw for example rectangle.*]]

- degree of nodes in C is nr

  (r digits are ruled out and there are a total of n digits)

- degree of nodes in D is $n - r$

  (each digit appears once in every row, so r times in total)

- thus graph is regular bipartite and so has perfect matching

- add row corresponding to matching

  [[*Select matching in drawn example and show it extends Latin rectangle.*]]

□