EECS 310: Discrete Math **Lecture 6**
Graph Theory

Reading: MIT OpenCourseWare 6.042 Chapter 5.3-5.5

Graph Coloring

Matching: edges represent compatibility.

Coloring: edges represent conflict.

[[*exam scheduling, coloring a map, updates to servers where certain pairs can't be taken down simultaneously since they cover similar critical functionality, etc.*]]

Def: *k-coloring*: label nodes with one of k colors so adjacent nodes get different colors.

Question: How many colors do you need for an n-node graph of type:

- empty graph
- line graph
- cycle, $n = 2k$
- "star" graph
- bipartite graph
- cycle, $n = 2k + 1$
- complete graph

Def: *chromatic number* $\chi(G)$: smallest k s.t. G is k-colorable.

[[*NP-hard to find smallest, if you can you get clay prize of \$1M. But can bound degree based on other properties. Ideas?*]]

Def: *clique number* $\omega(G)$: largest k s.t. G contains a complete graph on k vertices.

Claim: $\chi(G) \geq \omega(G)$

Def: *max degree* $\Delta(G)$: degree of max-degree vertex.

Claim: $\chi(G) \leq \Delta(G) + 1$

Proof: By induction.

[[*On what? Choices: max degree, nodes, edges.*]]

- By induction on number of vertices: $P(n)$ = for all k, a graph on n vertices with max deg. k is $(k+1)$-colorable.
- Base case: 1 vertex needs 1 color, so $(0+1) = 1$-colorable.
- Inductive hypothesis: n vertices with max deg. k is $(k+1)$-colorable.
- Inductive step:
  - Let G be an $(n+1)$ vertex graph with max deg at most k.
  - Remove v and incident edges to get G'. Color G' inductively.
  - Add back v and edges. Since v has at most k neighbors, must be an available color.

$\square$

Paths and Walks

Def: *walk*: sequence of vertices of G, $v_0, \ldots, v_k$, and edges of G, $\{v_0, v_1\}, \ldots, \{v_{k-1}, v_k\}$:

- may repeat vertices or edges
- *starts* at v_0
- *ends* at v_k
- *length* is k (number of times traverse an edge)

Def: *path*: walk where all v_i's are distinct

Example: Draw graph and specify a walk and a path.

Claim: If there's a walk, then there's a path.

Proof: well-ordering, see text

Note: length of path at most length of walk

Numbers of walks

[[*how many ways to get from here to there?*]]

Def: *adjacency matrix* A: $A_{ij} = 1$ if $\{i, j\} \in E$, 0 otherwise

Example: adjacency matrix of square with one diagonal

Claim: Number of walks of length k from i to j is ij'th entry of A^k.

Example: square and cube adjacency matrix of previous example

[[*proof gives insight to relationship between adjacency matrix multiplication and numbers of walks, important ingredient in pagerank.*]]

Proof: By induction on k.

- Base case, $k = 1$.
  - case 1: $\{i, j\} \in E$
    then $A_{ij} = 1$ and there's 1 walk from i to j of length one.
  - case 2: $\{i, j\} \notin E$
    then $A_{ij} = 0$ and there's no walk from i to j of length one.
- Inductive Step.
  - Let P^k_{xy} be number walks of length k from x to y.
  - By I.H., P^k_{tj} is tj'th entry of A^k, call it A^k_{tj}.
  - Group walks from i to j by first edge $\{i, t\}$:

$$P^{k+1}_{ij} = \sum_{t:\{i,t\}\in E} P^k_{tj}$$

  - Since $A_{it} = 1$ iff $\{i, t\} \in E$,

$$P^{k+1}_{ij} = \sum_{t=1}^{n} A_{it} P^k_{tj}$$

  - Using I.H.,

$$P^{k+1}_{ij} = \sum_{t=1}^{n} A_{it} A^k_{tj}$$

  - Which is A^{k+1}_{ij} by matrix multiplication.

Question: Find length of shortest paths?

[[*compute powers of A until* $A^k_{ij} > 0$]]

Connectivity

[[*can we get from here to there? can each node in the network send packets to each other node?*]]

Def: i, j *connected* if there is a path from i to j

Note: i connected to itself by convention

Def: G *connected* if all pairs of nodes connected

Example: Draw a disconnected graph, two edges and a triangle.

Def: *connected components*: subgraph of G consisting of a node and every node connected to it.

[[*Bounds on number connected components in graph G of n nodes?*]]

[[*Bounds on number of edges in connected graph G of n nodes? How about G with n connected components? 2 connected components?*]]

Claim: Every graph with n vertices and m edges has at least $n - m$ connected components.

Proof: Induction on m. Let $P(m) = \forall n \in \mathbb{N}, G$ with m edges has $n-m$ connected components.

- Base case: $m = 0$, each vertex is connected component so there are n connected components.
- Inductive step: Assume for every m-edge graph. Consider $(m+1)$-edge graph.
  - remove arbitrary edge $\{i, j\}$
  - by induction, remaining graph G' has $n - m$ connected components
  - add back edge
    * if i, j in same connected component of G', then G has $n-m > n-(m+1)$ connected components.
    * if i, j in different connected components of G', then two components merge when add back, so G has $n - m - 1 = n - (m+1)$ connected components.

[[*induction on edges and on nodes very common, we've seen both this lecture, questions?*]]

Note: In inductive step, take $(m+1)$-edge graph (or $(n+1)$-node graph) and *delete* element.

Build-up Error

Claim: If every node has degree ≥ 1, G is connected.

Question: Counterexample? [[*Graph of two edges*]]

Example: draw graph on two edges

Proof: Use induction on n.

- Let $P(n)$ = if every node in n-vertex graph has degree ≥ 1, then G is connected.
- Base case: only one 1-node graph, degree zero, statement vacuously true.
- Inductive step:
  - Consider n-node graph G' where each vertex has degree 1.
  - By assumption G' connected, i.e., there's a path between each i, j in G'
  - Now add one more node k with degree one to get $(n+1)$-node graph G. This node must connected to a node, say l, in G'.

– Then G is connected since for any $i, j \neq k$ we can use path from G' and for $i = k$, we can use edge $\{k, l\}$ with path from l to j in G'.

[[*Where's error? We can't get every $(n+1)$-node graph with min-degree 1 by adding to n-node graph with min-degree 1! Example above is counter-example.*]]

MANTRA:

SHRINK-DOWN, GROW-BACK.

[[*repeat out loud*]]

Note: if we shrink-down grow-back in above,

- Inductive step:
  - Consider $(n + 1)$-node graph G where each node has degree ≥ 1
  - Delete a node to get G'
  - Then in G' each node has degree $\geq$ ... uh oh!