EECS 310: Discrete Math **Lecture 7**
Graph Theory

Reading: MIT OpenCourseWare 6.042 Chapter 5.6

Review

- Matchings/perfect matchings
- Hall's Theorem: in a bipartite graph, L can be covered by a matching if $\forall S \subseteq L$, $|S| \leq |N(S)|$.
- Graph coloring:

$$\omega(G) \leq \chi(G) \leq \Delta(G) + 1$$

- Connectivity: G has at least $n - m$ connected components
- Induction on graphs: shrink down, grow back

[[*mini-quiz*]]

Cycles

Def: *closed walk*: walk with $v_0 = v_k$. **Def:** *cycle*: path with $v_0 = v_k$.

Claim: The following graph properties are equivalent:

1. bipartite
2. 2-colorable
3. no cycles with odd length
4. no closed walks with odd length

[[*useful since bipartite graphs come up a lot in practice, we will revisit this*]]

Proof:

- $1 \rightarrow 2$: use partition labels as colors
- $2 \rightarrow 3$: consider 2-coloring and any cycle $v_0, \ldots, v_k$.
  - since $\{v_i, v_{i+1}\} \in E$, v_i and v_{i+1} must have different colors
  - so $v_0, v_2, \ldots$ have one color and $v_1, v_3, \ldots$ have other color
  - since $v_0 = v_k$, they have same color
  - hence k is even
- $3 \rightarrow 4$: by contradiction
  - suppose odd closed walk and no odd cycle
  - let $v_0, \ldots, v_k$ be shortest odd closed walk
  - since no odd cycle, closed walk must repeat a vertex, say $v_i = v_j$
  - thus closed walk union of two other closed walks:

$$v_0, \ldots, v_i, v_{j+1}, \ldots, v_k$$

and

$$v_i, v_{i+1}, \ldots, v_j$$

which partition edge-traversals of original walk

  - since original is odd, one of these is odd, but shorter, contradiction

- $4 \to 1$: by contradiction
  - suppose no odd closed walks and not bipartite
  - not bipartite means some connected component G' not bipartite
  - consider any $v \in V'$ and for every $u \in V'$ define $dist(u)$ to be shortest path from u to v
  - define

$$L = \{u : dist(u) \text{ odd}\}$$

$$R = \{u : dist(u) \text{ even}\}$$

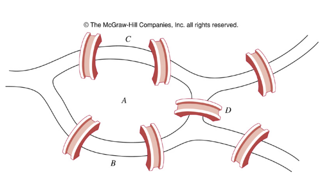
  - G' not bipartite means $\exists$ adjacent u_1, u_2 both in L or both in R
  - let P_i be shortest path from u_i to v
  - then P_1, P_2 either both odd (if $u_1, u_2 \in L$) or even
  - so closed walk consisting of union of P_1, P_2 and $\{u_1, u_2\}$ has odd length

[[*splicing/dicing arguments useful for path/cycle constructions*]]

Eulerian Paths and Tours

7 bridges of Konigsberg

[[*can you cross all 7 bridges without crossing any single bridge more than once?*]]



Def: *Eulerian walk*: use each edge exactly once.

Def: *Eulerian tour*: closed Eulerian walk.

[[*think of meter maid, wants to cover every street and not retrace steps*]]

Claim: A connected graph $G(V, E)$ has an Eulerian tour iff every vertex has even degree.

Proof:

- $\to$:
  - By direct proof.
  - Suppose G has Eulerian tour.
  - For every vertex $v \in V - \{v_0\}$, every time tour visits v, traverses two edges (one coming in, one going out).
  - Since each edge used exactly once and all edges are used, v has even degree.
  - For v_0, we can pair initial edge with final edge, so v_0 also has even degree.

- $\leftarrow$:
  - By direct proof.
  - Suppose G is connected and all vertices have even degree.
  - Construct tour as follows:
    * start from v_0 and continue traversing untraversed edges until tour gets stuck.
    * if there are untraversed edges, pick one $\{u, v\}$ with v on current tour and repeat first step with $v_0 = v$.
    * splice together tours.

  Claim 1: In first step, only get stuck at v_0.

* in any open walk W, endpoints have odd degree in W
* since all vertices have even degree, if we're not at v_0, we can continue

Claim 2: In second step, if $\exists$ untraversed edge, must be one with an endpoint on current tour.

* let $\{u_1, u_2\}$ be untraversed
* G is connected
* so exists path from v_0 to u_1
* earliest edge on path not on tour is adjacent to tour

□

Def: *Hamiltonian walk/tour*: visit each *vertex* exactly once.

[[*Much harder to find Hamiltonian paths/cycles, in fact NP-complete.*]]

[[*traveling salesman, visit each city exactly once and minimize distance, i.e., shortest hamiltonian path*]]