EECS 310: Discrete Math **Lecture 8**
Graph Theory

Reading: MIT OpenCourseWare 6.042 Chapter 5.8

Planar Graphs

Def: A graph is *planar* if it can be drawn in the plane without crossing edges.

[[*Useful for designing circuit boards, for example.*]]

Example: Draw planar graph, label faces

[[*cycles correspond to faces*]]

Note: bridges/dongles.

Recursive Defn

Def: *planar embedding*: non-empty set of closed walks (i.e. faces) constructed recursively by:

- **Base case**: single vertex v, one closed walk of length zero, one face.
- **Constructor case** (split a face): add edge $\{a, b\}$ between non-adjacent vertices on closed walk – splits face into two
- **Constructor case** (add a bridge): given two disconnected planar graphs G and H, add edge between vertices on faces – merges faces into one

[[*think of embedding on sphere, outer face not special*]]

Euler's Formula

Claim: Let G be a connected planar graph with n vertices, f faces, and m edges. Then $n - m + f = 2$.

Proof:

- base case: $m = 0$, $n = 1$, $f = 1$.
- constructor case (spit face): gain one face, gain one edge
- constructor case (add bridge): $m = m_G + m_H + 1$, $n = n_G + n_H$, $f = f_G + f_H - 1$, so

$$n - m + f =$$

$$(n_G+n_H)-(m_G+m_H+1)+(f_G+f_H-1) =$$

$$(n_G-m_G+f_G)+(n_H-m_H+f_H)-2 = 2$$

□

[[*Consequences: planar graphs can't have too many edges.*]]

Claim: A connected planar graph with n vertices and m edges has $m \leq 3n - 6$.

Proof: Let m_f be # edges on face f.

- Each edge is on at most 2 faces, so $\sum_f m_f \leq 2m$.
- Each face has at least 3 edges, so $\sum_f m_f \geq 3f$.
- Therefore $3f \leq \sum_f m_f \leq 2m$ so $f \leq \frac{2}{3}m$.

- By Euler, $2 = n - m + f \leq n - m + \frac{2}{3}m = n - \frac{1}{3}m$.
- Rearranging we get $\frac{1}{3}m \leq n - 2$ so $m \leq 3n - 6$.

$\square$

Example: Is K_4 planar?

$m = 4 \cdot 3/2 = 6$, $n = 4$, and $6 \leq 3 \cdot 4 - 6 = 6$, so maybe.

In fact, yes, draw embedding.

Example: Is K_5 planar?

$m = 5 \cdot 4/2 = 10$, $n = 5$, and $10 > 3 \cdot 5 - 6 = 9$, so no.

Example: Is $K_{3,3}$ planar?

In bipartite graph, each face has at least 4 edges! Replace the 3 by a 4 in proof.

- $4f \leq \sum_f m_f \leq 2m$ so $f \leq \frac{1}{2}m$
- $2 = n - m + f \leq n - m + \frac{1}{2}m = n - \frac{1}{2}m$ or $m \leq 2n - 4$

$m = 3 \cdot 3 = 9$, $n = 6$, and $9 > 2 \cdot 6 - 4 = 8$, so no.

Subclaim: A graph containing K_5 or $K_{3,3}$ is not planar.

Def: *graph minor*: graph obtained from G by deleting edges/vertices and merging vertices.

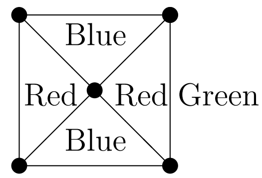
Subclaim: A graph is not planar iff it contains K_5 or $K_{3,3}$ as a minor.

Coloring Planar Graphs

A 5-color theorem.

Goal: 5-color a map

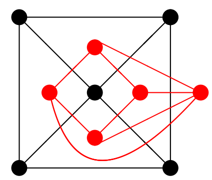
Example:



Duality:

- vertex for each face
- edge if faces share a boundary

Example:



Note: Dual is also planar.

Claim: Any planar graph has a vertex of degree at most 5.

Proof: Suppose not. Then $2m$ = sum of degrees $\geq 6n$, so $m \geq 3n$, but by Euler, $m \leq 3n - 6$. $\square$

Claim: Planar graphs can be 6-colored.

Proof:

- By induction on vertices.
- Inductive step: Remove vertex v with degree at most 5, inductively color graph, add back vertex and color with a color not equal to its neighbors.

$\square$

Claim: Planar graphs can be 5-colored.

Proof: As above. Note v has two neighbors u and w that are not themselves neighbors (otherwise we found a K_5). Merge u and w

and color graph. Split them, now v's neighbors have just 4 colors! $\square$