EECS 310, Fall 2011
Instructor: Nicole Immorlica
Problem Set #2
Due: October 4, 2011

1. (25 points) Clarification: The notation $p|n$ means that p divides n (meaning that p is a divisor of n).

   Our goal here is to show that for any prime p and natural numbers n, m, if $p|nm$, then $p|n$ or $p|m$. We shall do this by using proof by least-counter-example. Note, there are three variables of interest here, and so we will need to apply the well-ordering principle *twice*.

   Let $P(p, n, m)$ denote the predicate that if $p|nm$ then $p|n$ or $p|m$. Assume that the theorem is false and let

   $$F = \{(p, n, m)|P(p, n, m) \text{is false}\}$$

   By our assumption that this theorem is false, F is a non-empty set. Using the well-ordering principle let p' be the smallest p contained in any $(p, n, m) \in F$. Using the well-ordering principle again, let n' be the smallest integer such that $(p', n', m) \in F$. That is n' is the smallest integer that is paired with the smallest prime p' such that $P(p', n', m)$ is false (note the choice here of m doesn't matter). So this means that $p'|n'm$ is true but $p'|n'$ is false and $p'|m$ is false. We now will prove several facts about n'.

   (a) (1 point) Prove that $n' \neq 0$ and $n' \neq 1$

   (b) (8 points) Prove that n' is prime.

   (c) (8 points) Prove that $n' < p'$.

   (d) (8 points) Using these three above facts, prove this theorem. [Hint: Our assumption was that p' was the smallest prime number that violated the predicate. Since we know that $n' < p'$ and n' is prime, then n' cannot violate the predicate. Use this to derive a contradiction with our assumptions]

2. (25 points) The *factorial* of a number n, written $n!$, is the product of all integers from 1 to n, i.e.,

   $$n! = \prod_{i=1}^{n} i.$$

   Consider the following proof by induction. Let $P(n)$ be the predicate that $n!$ has n distinct divisors $d_1, \ldots, d_n$ that sum to $n!$. We want to show that $P(n)$ is true for all $n \geq 3$.

Part (a).

Consider the following proof by induction.

a.1) (2 points) Prove the base case $P(3)$.

Assume the inductive hypothesis $P(n)$. Then, for the inductive step $P(n) \rightarrow P(n+1)$, show that:

a.2) (5 points) the numbers $(n+1)d_1, \ldots, (n+1)d_n$ divide $(n+1)!$ (where $d_1, \ldots d_n$ are the divisors of $n!$),

a.3) (5 points) and these numbers sum to $(n+1)!$.

a.4) (2 points) Why have we not yet proved the claim?

Part (b).

The problem in the above proof can be fixed by strengthening the inductive hypothesis. It would be nice if we could argue that both d_1 and nd_1 divide $(n+1)!$.

b.1) (1 points) Can you think of a value for d_1 for which this argument becomes obvious? (If you can not, email the TAs for the solution to this part.)

b.2) (10 points) Write the complete proof of the theorem using the strengthened inductive hypothesis.

3. (50 points) The *Fibonacci numbers* are a sequence of numbers defined as follows:

- $F_0 = 0$,
- $F_1 = 1$,
- $\forall n > 1, F_n = F_{n-1} + F_{n-2}$.

As indicated by this xkcd comic,

the first few Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, \ldots$. This sequence has surprisingly many elegant properties, some of which we prove here. Use proof by induction or strong induction to show the following.

(a) (10 points) The sum of any 10 consecutive Fibonacci numbers is divisible by 11.

(b) (15 points) Any number can be written in a *distinct* way as the sum of Fibonacci numbers. This means any number n can be written as $n = \sum_i a_i F_i$ where F_i is the ith Fibonacci number and a_i is either 0 or 1. In words, this means that I can express any number n as the sum of some Fibonacci numbers, without having to repeat any Fibonacci number.

(c) (10 points) For any $n \geq 0$ and $m \geq 1$, $F_{m+n} = F_m \cdot F_{n+1} + F_{m-1} \cdot F_n$.

(d) (15 points) Prove that the n'th Fibonacci number $F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}$ where $\phi = \frac{1+\sqrt{5}}{2}$. ϕ is a special number in mathematics, known as the golden ratio. There are many interesting properties and occurrences of the golden ration, one of them being that it expresses the ratio of the number of branches along the stems in many plants and trees. Also, the following properties may be of interest...

$$\phi^2 = (\frac{1+\sqrt{5}}{2})^2 = \frac{1+2\sqrt{5}+5}{4} = 1 + \frac{1+\sqrt{5}}{2} = 1 + \phi$$

[Hint: If you are stuck, you should look for a similar identity to the one above to help you through this problem. It is not hard to derive for yourself but it may require playing around with it]

4. **Challenge Problem!** Students that are enrolled in the Arts High School are well known for being extremely fashionable. A new trend is introduced this season for the students to wear hot pink shoes. Students in the class are sitting in a $n \times n$ grid. Since this new trend is quite bizarre only the most fashionable students adopt it. The rest students will only follow if two or more of their neighbors in the class (among the four possible: left, right, back, and front but not diagonal) are already following it. Consider the following example where $n = 6$ and students that wear hot pink shoes are marked with $\times$.

$\times$				$\times$	
	$\times$				
		$\times$	$\times$		
		$\times$			
			$\times$		$\times$

This new fashion spreads every day. We assume that once a student starts to wear hot pink shoes, he or she will never stop doing so.

In this example, over the next few time-steps, all the students in class will adopt this trend.

$\times$				$\times$	
	$\times$				
		$\times$	$\times$		
		$\times$			
			$\times$		$\times$

$\Longrightarrow$

$\times$	$\times$			$\times$	
$\times$	$\times$	$\times$			
	$\times$	$\times$	$\times$		
		$\times$			
		$\times$	$\times$		
		$\times$	$\times$	$\times$	$\times$

$\Longrightarrow$

$\times$	$\times$	$\times$		$\times$	
$\times$	$\times$	$\times$	$\times$		
$\times$	$\times$	$\times$	$\times$		
	$\times$	$\times$	$\times$		
		$\times$	$\times$	$\times$	
		$\times$	$\times$	$\times$	$\times$

Prove the following: If fewer than n students in class are initially wearing hot pink shoes, the whole class will never adopt the trend.

Hint: When one wants to understand how a system such as the above evolves over time, it is usually a good strategy to (1) identify an appropriate property of the system at the *initial stage, and (2) prove, by induction on the number of time-steps, that the property is preserved at every time-step. So look for a property (of the set of the students following the trend) that remains invariant as time proceeds.*