EECS 310, Fall 2010
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Problem Set #2
Due: October 7, 2010

1. (24 points) Functions
   (a) (4 points) Give an example of a function from $\mathbb{N}$ to $\mathbb{N}$ that is
      1. injective but not surjective
      2. surjective but not injective
      3. bijective (but not the identity function)
      4. neither injective nor surjective
   (b) (20 points) Suppose $g : A \to B$ and $f : B \to C$. Then the *composition* of f and g, denoted $f \circ g$, is a function from A to C defined by $f \circ g(a) = f(g(a))$.
      1. If f and $f \circ g$ are both injective, must g be injective? Justify your answer.
      2. If f and $f \circ g$ are both surjective, must g be surjective? Justify your answer.

2. (24 points) Sets

   This problem demonstrates the limitations of computers. In particular, we say a function is *computable* if there is a computer program that finds the values of the function on any input. Using counting arguments, we can prove that there are functions that are not computable.
   (a) (12 points) Prove that the set of all computer programs that can be written using symbols from a finite alphabet (like, say, English) is countable. (Note: A computer program is a finite string of symbols from the alphabet.)
   (b) (12 points) Prove that the set of all functions from $\mathbb{N}$ to $\{0, 1\}$ is uncountable.

3. (24 points) Induction, warmup

   Use proof by induction to show the following.
   (a) (8 points) For any integer $n \geq 1$, 6 divides $n^3 - n$.
   (b) (8 points) For any integer a and integer $n \geq 1$, $a^n - 1$ is divisible by $(a - 1)$.
   (c) (8 points) For any integer $n \geq 1$, $2^n > n$ (in general, the *exponential* functions grow much faster than the *polynomial* functions, which is why we care so much about polynomial-time algorithms).

4. (28 points) Induction, harder
   (a) (14 points) A set of n elements has 2^n subsets (hence when searching an input space algorithmically for a subset of elements that satisfy a particular property, brute-force search often takes too long).
   (b) (14 points) Show that for any $n \geq 1$, there exists an integer with n digits from $\{1, 2\}$ that is divisible by 2^n. For example, for $n = 4$, 2112 is a four-digit number with digits from $\{1, 2\}$ that is divisible by 2^4.