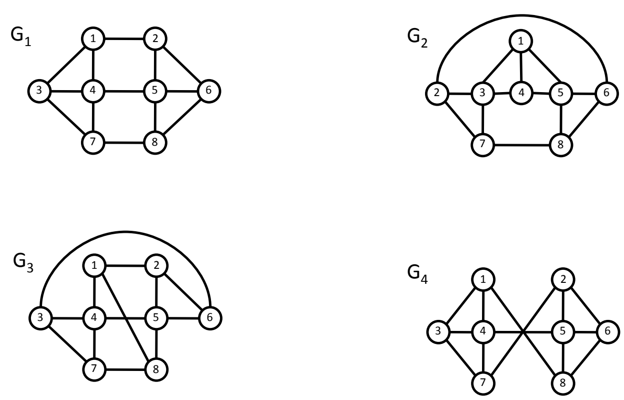
EECS 310, Fall 2011
Instructor: Nicole Immorlica
Problem Set #3
Due: October 13, 2011

1. (25 points) This question explores some basic properties of graphs.
   (a) (12 points) Prove the following claim *using induction.* Consider carefully whether it is easier to do induction on the number of vertices or the number of edges.

       The number of vertices in a graph of odd degree is even.

   (b) (12 points) There is an EECS faculty party with N professors going. At this party, certain pairs of professors are already friends (friendships are mutual i.e. if Alice is friends with Bob then Bob must be friends with Alice), while other professors are not friends (for this example, people are not friends with themselves). For a particular guest g_i at this party, let $F(g_i)$ be the number of friends that g_i has at this party. Prove that there are at least two professors that have the same number of friends at this party. In notation, prove that $F(g_i) = F(g_j)$ for some $i \neq j$.

2. (25 points) Determine which of the following graphs are isomorphic. If two graphs are isomorphic, describe an isomorphism. If they are not, give a property that is preserved under isomorphism that one graph satisfies and the other does not.



3. (25 points) For each of the below statements, prove the claim or provide a counterexample.
   (a) (5 points) A line graph is bipartite.
   (b) (5 points) A cycle is bipartite.
   (c) (15 points) A *hypercube* is a graph in which the vertices are binary bit-strings of length k and two vertices are connected if and only if they differ in exactly one bit. A hypercube is bipartite.

4. (25 points) Recall the puzzle from the first lecture where we wanted to tile a chessboard with dominos. We argued that a chessboard that is missing the bottom-left and top-right square can not be tiled with dominos (where each domino occupies exactly two adjacent positions on the chessboard). Here we will use Hall's theorem to generalize this puzzle.

   (a) (15 points) Given a chessboard with missing squares, show how to construct a bipartite graph that has a perfect matching if and only if the chessboard can be tiled with dominos. Be sure to explain your construction (i.e., how to convert from the perfect matching to the tiling and vice versa).

   (b) (10 points) Based on this construction and Hall's theorem, state a necessary and sufficient condition for a chessboard to be tilable with dominos. Don't use graph terminology (matching, node, edge, etc.), but rather aim for an explanation that could be published in the popular press.