EECS 310, Fall 2011
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Problem Set #4
Due: October 20, 2011

1. (20 points) You have n jobs $J = \{1, \ldots, n\}$ that must be scheduled on m machines $M = \{1, \ldots, m\}$. Each job i can only be processed by a subset $M_i \subseteq M$ of the machines. Furthermore, each machine can process only a total number of k_i jobs. Derive a condition similar to that in Hall's Theorem to determine whether all the jobs can be scheduled. Use Hall's Theorem to prove that your condition is necessary and sufficient.

2. (20 points) A company wishes to interview n candidates for job opening. Each candidate can be interviewed by exactly k managers and, conversely, each manager is qualified to interview exactly k candidates. As CEO, you must find a schedule of interviews such that:

   - each interview takes one hour,
   - each manager interviews at most one candidate per hour,
   - each candidate is interviewed at most once in any given hour,
   - and each candidate is interviewed k times.

   Prove that there exists a schedule of interviews such that the entire interview process takes k hours.[1]

3. (20 points) Prove or disprove the following statement: for some $n \geq 3$ (n boys and n girls for $2n$ total people), there exists a strict set of preferences such that every matching is stable.

4. (20 points) Consider a marriage setting with m men and $4m$ women, suppose each person has strict preferences over members of the opposite sex, and suppose that every man must marry 4 women. A set of marriages is stable if there does not exist a man m and a woman w such that w is not one of m's wives and w prefers m to the man that she is married to and m prefers w to at least one of the women he is married to. In other words, there does not exist a scenario where a woman wants to run away from her husband to a man (because she likes him more) who wants to kick out one of his wives for this new one (because he likes her more). Prove that for any preferences, there exists a set of stable marriages. You may assume the theorem proven in lecture that the courtship ritual produces a stable matching.

5. (20 points) Draw the following graphs.

   (a) (10 points) A connected graph on 4 vertices whose chromatic number is less than its maximum degree. Clearly indicate a coloring that uses less colors than the maximum degree.

[1] Hint: use Hall's Theorem plus induction on an appropriately-defined bipartite graph.

(b) (10 points) A connected graph on 4 vertices whose chromatic number is more than its maximum degree. Clearly indicate a coloring and explain why you needed as many colors as you used.

6. **Challenge Problem!**

   There are a n ghostbusters and n ghosts in a large, flat field. Each ghostbuster needs to zap exactly one ghost. However, as we all should know, you are not allowed to cross the beams (a beam is a straight line segment connecting a ghostbuster to a ghost). Prove that no matter where the ghosts and ghostbusters are standing in this field, it is always possible to find an assignment where each ghostbuster zaps exactly one ghost and each ghost is zapped by exactly one ghostbuster and none of the beams cross (both the ghosts and ghostbusters are on the ground, so the beams cannot be shot over one another).
   [2]

[2]Hint: this puzzle has an elegant solution using one of the four proof recipes from the beginning of the quarter (and it has nothing to do with matching theory).