EECS 310, Fall 2011
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Problem Set #5
Due: October 27, 2011

1. (25 points) Let $(s_1, s_2, ..., s_n)$ be an arbitrarily distributed sequence of the numbers $1, 2, ..., n-1, n$ (i.e., a permutation). For instance, for $n = 5$, one arbitrary sequence could be $(5, 3, 4, 2, 1)$. Define the graph $G = (V, E)$ as follows:

   1. $V = \{v_1, v_2, ..., v_n\}$

   2. $\{v_i, v_j\} \in E$ if either:
   - $j = i + 1$, for $1 \leq i \leq n-1$
   - $i = s_k$, and $j = s_{k+1}$ for $1 \leq k \leq n-1$

   (a) Prove that this graph is 4-colorable for any $(s_1, s_2, ..., s_n)$.[1]

   (b) Suppose $(s_1, s_2, ..., s_n) = (1, a_1, 3, a_2, 5, a_3, ...)$ where $a_1, a_2....$ is an arbitrary distributed sequence of the even numbers in $1, ..., n$. Prove that the resulting graph is 2-colorable.

2. (25 points) An open Eulerian walk in a graph is a walk that traverses each edge exactly once and returns to a different vertex than it started from. Prove that a connected graph has an open Eulerian walk if and only if it has exactly two vertices of odd degree.[2]

3. (25 points) Let G be a graph that has exactly $2k$ vertices of odd degree. Show that there are k edge-disjoint paths each of which joins a different pair of vertices of odd degree.[3]

4. (25 points) On planet XY live aliens whose alphabet contains only two letters, x and y. There are 2^d cities on this planet, each with a distinct name consisting of d characters. The government has designed a public transportation system that allows aliens to teleport from one city to another if the names differ in just one letter. Alien Gorlack would like to take a world tour. Show that for any d, starting from his hometown, Gorlack can teleport around the world visiting each city exactly once and return to his hometown.

[1]Hint: Use the fact that a line graph is 2-colorable.

[2]There is an elegant solution for the proof of the converse that uses the theorem statement from class about the necessary and sufficient conditions for existence of an Eulerian tour.

[3]HINT: Use a strengthened inductive hypothesis. For the base case, use the fact proved in the previous problem that if a graph has exactly two vertices of odd degree, then it has an Eulerian path.