EECS 310, Fall 2011
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Problem Set #6
Due: November 8, 2011

1. (25 points) Consider the following false claim:

   Let G be a simple graph with maximum degree at most k. If G has a vertex of degree less than k, then G is k-colorable.

   (a) (5 points) Give a counterexample to the false claim when $k = 2$.

   (b) (15 points) Consider the following proof of the false claim:

   Proof. Proof by induction on the number of vertices n.

   - **Inductive Hypothesis**: $P(n)$ is defined to be: Let G be a graph with n vertices and maximum degree at most k. If G has a vertex of degree less than k, then G is k-colorable.
   - **Base Case**: $(n = 1)$ G has only one vertex and is 1-colorable, so $P(1)$ holds.
   - **Inductive Step**: We may assume $P(n)$. To prove $P(n+1)$, let G_{n+1} be a graph with $n + 1$ vertices and maximum degree at most k. Also, suppose G_{n+1} has a vertex v of degree less than k. We need only prove that G_{n+1} is k-colorable.
     To do this, first remove the vertex v to produce a graph G_n with n vertices. Removing v reduces the degree of all vertices adjacent to v by 1. So in G_n, each of these vertices has degree less than k. Also the maximum degree of G_n is at most k. So G_n satisfies the conditions of the inductive hypothesis $P(n)$. We conclude that G_n is k-colorable.
     Now a k-coloring of G_n gives a coloring of all vertices of G_{n+1} except for v. Since v has degree less than k, there will be fewer than k colors assigned to the nodes adjacent to v. So among the k possible colors, there will be a color not used to color these adjacent nodes, and this color can be assigned to v to form a k-coloring of G_{n+1}.

   Identify the exact sentence where the proof goes wrong and explain the error.

   (c) (5 points) Add a condition to the theorem statement to make it a true claim, and explain how to revise the above proof to give a proof of this true claim.

2. (40 points) A face-coloring of a planar graph is an assignment of colors to the faces such that no two adjacent faces have the same color.Consider a planar graph G that has no bridges or dongles, i.e., every edge is on a cycle or, equivalently, every edge is adjacent to two distinct faces.

   (a) (20 points) Show that if the faces of G can be colored with two colors, then it has an Euler tour.

   (b) (20 points) Show that the converse also holds, i.e., if G has an Eulerian tour, then the faces can be 2-colored.

3. (35 points) Suppose that $d_1, d_2, ..., d_n$ are n positive integers such that their sum is $2n-2$. Show that there is a tree that has n vertices so that the degrees of these vertices are $d_1, d_2, ...d_n$.

4. **Challenge Problem!**

   Abandon all hope, ye who enter here

   One day, you find yourself in the unfortunate position of having to bargain with the devil for your soul (it happens to all of us at some point). The devil proposes that you two play the following game for your soul (after that whole mess in Georgia, he now prefers mathematical games to fiddle showdowns).

   There is a perfectly circular table of radius R. Both you and the devil have an infinite stack of gold coins, each of which is a circle with radius S. Assume that R is much greater than S (this actually doesn't make too much of a difference so long as R is greater than S, but assuming that the coins are very small may help you). You take turns placing coins on this table such that

   1. The entire coin rests upon the table (i.e. the coin is contained in the circle that is the table).

   2. No two coins overlap (that is they do not rest upon each other but their sides may touch).

   The first person who cannot place a coin in a valid manner is the loser.
   Lucifer, the clever devil that he is, insists upon going first. Why does he do this? That is, prove that the first player has a strategy that guarantees himself a victory, regardless of how good the second player is.