EECS 310, Fall 2011
Instructor: Nicole Immorlica
Problem Set #7
Due: November 15, 2011

1. (20 points) A k-ary tree is a tree in which each internal node has at most k children. The *height h of a node* is the length of the path from the root to the node. The *height H of a tree* is the maximum height of any node.
   (a) (10 points) Derive a formula for the maximum number of nodes of height h in a k-ary tree. Prove that your formula is correct using induction.
   (b) (10 points) Using the above, write a summation to compute the maximum number of nodes in a ternary tree of height at most H and compute the closed form.

2. (20 points) Consider an $n \times n$ grid.
   (a) (5 points) How many squares does the grid contain? For here, you do not give to a true closed-form solutions, your solution may include summations (it probably should).
   (b) (5 points) How many rectangles does the grid contain?
   (c) (10 points) Suppose the grid is missing an $(\frac{(n-k)}{2} \times \frac{(n-k)}{2})$-sized piece from each of its corners (so it now looks like a cross in which each board is $n \times k$. Also, you can assume that $\frac{n-k}{2}$ is an integer). How many rectangles does it contain now?

3. (20 points) Give combinatorial proofs of the following identities.
   (a) (10 points) $\binom{3n}{2} = \binom{n}{2} + 2n^2 + \binom{2n}{2}$
   (b) (10 points) $\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$
   Hint: Given $2n$ people – n boys and n girls – count the number of ways to form a committee of n people such that the chair of the committee is a girl.

4. (20 points) Let S be an n-element set and let $S_1, \ldots, S_n$ be subsets of S such that $|S_i \cap S_j| \leq 1$ for all $1 \leq i < j \leq n$. We will prove by contradiction that at least one set S_i has size at most $2\sqrt{n}$. Suppose every set has size at least $2\sqrt{n}$ and yet the sizes of the intersections are all small (at most 1).
   (a) (10 points) Use the pigeonhole principle to show that this implies some element is in at least $2\sqrt{n}$ sets.
   (b) (10 points) Argue that among the $2\sqrt{n}$ sets from the previous part, there must be two whose intersection has cardinality at least 2.

5. (20 points) Suppose $\Pr\{.\} : S \to [0, 1]$ is a probability function on a sample space S and let B be an event such that $\Pr\{B\} > 0$. Define a function $\Pr_B\{.\}$ on outcome $w \in S$ by the rule: $\Pr_B\{w\} = \{\Pr\{w\}/\Pr B \text{ if } w \in B; 0 \text{ if } w \notin B\}$.
   (a) (10 points) Prove that $\Pr_B\{.\}$ is also a probability functio on S according to Definition 14.4.2 in the MIT notes.

(b) (10 points) Prove that

$$\Pr_B\{A\} = \frac{\Pr\{A \cap B\}}{\Pr\{B\}}$$

for all $A \subseteq S$.

6. **Challenge Problem.** Consider a large supply of trucks which have the following properties: each truck has only a 1-gallon tank of gas; the mile-per-gallon for each truck is exactly 1; and gas can be transfered between trucks without any loss. At the beginning, all trucks are gathered at one side of a wide desert, and our task is to deliver an extremely valuable item to the other side of the desert. Show that with a large enough number of trucks, no matter how wide the desert is, it is always possible to finish the task. HINT: Use the fact that for any d there is an n such that the n'th harmonic number H_n is greater than d.