EECS 310, Fall 2011
Instructor: Nicole Immorlica
Problem Set #8
Due: November 22, 2011

1. (30 points) Let $S = \{1, \ldots, n\}$ and let π be a random permutation on S (i.e., $\pi : S \rightarrow S$ is a random bijection). Let $\pi(k)$ represent the k^{th} element in the permutation π.

   For a subset A of S let $f(A)$ be the minimum of $\pi(a)$ over all elements of $a \in A$. For example, if $n = 10$, $A = \{2, 5, 7\}$, and $\pi(2) = 3, \pi(5) = 10, \pi(7) = 2$, then $f(A) = 2$. Let A and B be two arbitrary subsets of S. Express the following in terms of A, B, and S. Be sure to explain your calculations.

   (a) (6 points) What is the probability that $\pi(1) = 1$?

   (b) (12 points) For an arbitrary element $a \in A$, what's the probability that $f(A) = \pi(a)$?

   (c) (12 points) What is the probability $f(A) = f(B)$?

2. (25 points) Consider the following hat puzzle: Three players enter a room wearing a hat. Each hat is either red or blue, and the color of a hat is chosen uniformly at random. A player can see every hat but his own. Each player must write on a card either *red*, *blue*, or *pass*. After everyone writes down something, the cards are simultaneously revealed (so no helpful information is revealed to the other players by what is written down on the card). If at least one player writes the color of his/her hat, and no player writes an incorrect color, the players win. Players can talk and coordinate a strategy before entering the room, but once they are in the room, no communication is allowed. Devise a strategy that yields strictly better than 1/2 probability of winning. Clearly state your strategy and calculate the probability that players win with your strategy using the four-step method discussed in lecture.

3. (20 points) Upon hearing that your mom just baked two apple pies, you rush home hoping to get a piece before your greedy little brother eats it all. Assume that:

   - it is equally probable that 2, 1, or zero pies are left by the time you get home (1/3 probability of each), and
   - if both pies are left, the probability you smell pie when you walk in the front door is 1; if only one is left it's 2/3; if they're both gone it's 1/3 (the smell lingers).

   Compute the following probabilities, explaining your calculation in each step.

   (a) (5 points) What is the probability there is no pie left and you still smell pie when you walk in?

   (b) (5 points) What is the probability you'll smell pie when you walk in?

   (c) (10 points) What is the probability there is at least one pie left given you smell pie when you walk in?

4. (25 points) Consider n tosses of a coin whose probability of a fair coin. Calculate the following, making sure to explain each step.

   (a) (8 points) Let X be a random variable equal to the number of tosses until you first encounter the pattern HH. (Count HH among the tosses; hence if the outcome is $HTHTHHHT$, then $X = 6$.) What is $E[X]$?

   (b) (8 points) Let Y be a random variable equal to the number of tosses until you first encounter the pattern HT. (Count HT among the tosses; hence if the outcome is $HTHTHHHT$, then $X = 2$.) What is $E[Y]$?

   (c) (5 points) Are X and Y independent random variables? Explain your answer.

   (d) (4 points) Consider the following bet: a coin is flipped until either HT or HH appears. If HT appears first, you win a dollar. Otherwise, if HH appears first, you lose a dollar. Is this a fair bet? Explain your answer.

5. **Challenge Problem.** A family has $2n$ children named $k_1, \ldots, k_{2n}$. It's Christmas time and each child k_i has hung a carefully labeled stocking i from the mantle. Santa arrives on Christmas Eve with $2n$ gifts $g_1, \ldots, g_{2n}$ where gift g_i is intended for kid k_i. Unfortunatley, Santa can't read in the dark and so deposits the gifts into the stockings in a random order. The goal of the children is to figure out which stocking contains their gift. They enter the room one by one and are allowed to look inside the stockings but can not remove gifts from stockings. Furthermore, they are not so patient and therefore only look in at most n stockings before giving up, after which they go off to breakfast without communicating their findings to the other siblings. Let E_i be the event that the i'th child finds his gift. Describe a search strategy for the children maximizing the probability that *every single child finds his gift* (any strategy with a constant probability will be accepted; the optimal probability is around 1/3). Note that if each child simply selects a random n boxes, the probability they all find their gifts is $(1/2)^n$ which approaches zero as n grows, so finding a constant here is quite impressive.[1]

[1] HINT: the strategy must be adaptive, i.e., the second stocking that you search should be a function of what you saw in the first one.