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Geometric phase for a dimerized disordered continuum: Topological shot noise

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Abstract. – Geometric phase shift associated with an electron propagating through a dimerized disordered continuum is shown to be 0, or $\pm\pi$ (modulo 2π), according to whether the associated circuit traversed in the two-dimensional parameter space excludes, or encircles a certain singularity. This phase shift is a topological invariant. Its discontinuous dependence on the electron energy and disorder implies a statistical spectral and conductance fluctuation in a corresponding mesoscopic system. Inasmuch as the fluctuation derives from the discreteness of the phase shift, it may aptly be called a topological shot noise.

The geometric phase, that is the phase shift acquired by a non-degenerate eigenstate of a quantum-mechanical Hamiltonian as the latter is cycled adiabatically through a closed circuit in the space of its parameters, is now a well-known and well-studied anholonomy in quantum physics [1]. This phase depends on the geometry of the circuit traversed, and not on how it is traversed. Hence, the name geometric phase, as opposed to the dynamical phase. It was re-discovered and discussed explicitly in the modern context by Berry [2], (hence also known as the Berry phase after him) who had envisaged it for a dynamical sub-system of interest with *fast* degrees of freedom, coupled to and enslaved by another sub-system of relatively *slower* degrees of freedom (the *parameters*), with the total system wave function remaining single-valued, of course. The idea has since been generalized to the case of non-adiabatic evolution [3], with the only condition that the evolution be confined to the same Hilbert space. The geometric phase in general varies continuously with the geometry of the parametric circuit traversed.

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Thus, *e.g.*, in the case of a spin-(1/2) object (an electron, say) in an external magnetic-field vector cycled over a cone, the geometric phase acquired by the spin eigenstates is $\pm 1/2$ times the conical solid-angle so subtended [2]. There are, however, interesting exceptions where the geometric phase can be a topologically invariant quantity having only discrete allowed values. A case in point is the Zak [4] phase acquired by the Bloch wave for a one-dimensional one-band periodic system where the parameter, namely the wave vector $\vec{q}$, is cycled over the closed reciprocal space (the first Brillouin zone), with $\vec{q}$ and $\vec{q}+\vec{G}$ identified. (Here $\vec{G} \equiv$ shortest reciprocal lattice vector). In the present work, we report yet another case of a topological phase shift that appears naturally when we consider an electron propagating in a dimerized disordered one-dimensional lattice, treated here in the continuum limit. Here, the time-independent Schrödinger equation for the wave amplitude can be transformed into the *spatial* evolution equation for a pseudo-spin-(1/2) object in a pseudo-magnetic field generating a two-dimensional parameter space. The evolution, however, turns out to be *non-unitary*. Following the recently developed procedure for non-unitary evolution [5,6], we have calculated the geometric phase for a circuit and found it to be 0, or $\pm\pi$ according to whether the circuit traversed excludes, or encircles a certain singular point in the parameter space. This topological phase depends discontinuously on disorder and on the electron energy, and its discreteness is, therefore, expected to introduce a shot noise characteristic in the statistics of spectral and conductance fluctuations for certain mesoscopic systems. Observable and calculable consequences for disordered mesoscopic systems are discussed.

Consider the Schrödinger equation for the wave amplitude ψ_n for a 1D tight-binding system given by

$$i\hbar\frac{\partial\psi_n}{\partial t} = V_n^{\mathrm{E}}\psi_n - \nu(\psi_{n+1}+\psi_{n-1}), \tag{1a}$$

$$i\hbar\frac{\partial\psi_n}{\partial t} = V_n^{\mathrm{O}}\psi_n - \nu(\psi_{n+1}+\psi_{n-1}), \tag{1b}$$

where dimerized disorder has been introduced through the random site-diagonal potentials V_n^{E} and V_n^{O} for the even and the odd sites, respectively. More specifically, we set $V_n^{\mathrm{E}} = U_n + \eta_n$, $V_n^{\mathrm{O}} = U_n - \eta_n$ and treat U_n and η_n as statistically independent random variables. This bipartite structure is essential to our work. We now proceed to the continuum limit, but without losing the two-sublattice structure. For this, we define [7]

$$\psi_{2n}(t) = i^{2n}(2s)^{1/2}\psi_{\mathrm{E}}(x,t), \tag{2a}$$

$$\psi_{2m+1}(t) = i^{2m+1}(2s)^{1/2}\psi_{\mathrm{O}}(y,t), \tag{2b}$$

and let the lattice spacing $s \to 0$, total site number $N \to \infty$, keeping the total length $L = Ns$ fixed and setting $2ns = x$, $(2m+1)s = y$. In order to have a well-defined continuum limit, the coupling ν must also be scaled as $\nu \to \infty$, keeping $2s\nu/\hbar = v_{\mathrm{F}}$. Then, defining $\psi_{\mathrm{L}} = (\psi_{\mathrm{E}} + \psi_{\mathrm{O}})/\sqrt{2}$ and $\psi_{\mathrm{R}} = (\psi_{\mathrm{E}} - \psi_{\mathrm{O}})/\sqrt{2}$, we get

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\psi_{\mathrm{L}}\\ \\ \psi_{\mathrm{R}}\end{pmatrix} = \begin{pmatrix} U(x) - i\hbar v_{\mathrm{F}}\dfrac{\partial}{\partial x} & \eta(x) \\ \\ \eta(x) & U(x) + i\hbar v_{\mathrm{F}}\dfrac{\partial}{\partial x}\end{pmatrix}\begin{pmatrix}\psi_{\mathrm{L}}\\ \\ \psi_{\mathrm{R}}\end{pmatrix}. \tag{3}$$

It is clear that the above equation describes the left-going and the right-going wave amplitudes (also called the left- and right-movers in the context of a two-component Dirac equation for a massless particle in 1-space + 1-time dimension) with the back scattering admixing them. Thus, for $U(x) = 0 = \eta(x)$, the electron energy $E = \pm\hbar v_{\mathrm{F}}k$, where k is the wave vector. We

can rewrite it by introducing a two-component wave amplitude ψ as

$$\psi = \begin{pmatrix} \psi_{\mathrm{L}} \\ \\ \psi_{\mathrm{R}} \end{pmatrix}, \tag{4}$$

and the evolution equation (3) then becomes

$$i\hbar v_{\mathrm{F}} \frac{\partial \psi}{\partial x} = -(E - U(x))\sigma_z \psi + i\eta(x)\sigma_y \psi \equiv H_{\mathrm{eff}} \psi, \tag{5}$$

where we have set $i\hbar \frac{\partial}{\partial t} \equiv E$, the electron energy. This is our main equation and can be studied for possible geometric phases. In this form the connection with a pseudo-spin 1/2 evolving spatially in a pseudo-field is clear. However, the evolution is now in *space* and is *non-unitary* because of the imaginary "i" occurring on the right-hand side of the equation.

We now proceed to calculate the geometric phase associated with the spatial evolution given by eq. (5). First note that the instantaneous eigenvalues of the evolution (non-Hermitian) Hamiltonian H_{eff} in eq. (5) are

$$\lambda_{\pm} = \pm\sqrt{(E - U(x))^2 - \eta(x)^2} \tag{6}$$

and the corresponding right-eigenvectors (unnormalized) are

$$|\pm\rangle = \begin{pmatrix} \dfrac{(E - U(x)) \mp \sqrt{(E - U(x))^2 - \eta(x)^2}}{\eta(x)} \\ \\ 1 \end{pmatrix}, \tag{7}$$

with bi-orthogonality between the left- and the right-eigenvectors.

Consider now the 1D disordered dimerized system as terminated asymptotically into two ordered leads that may be joined. We now consider the evolution of the two-component wave amplitude $\psi(x)$ as x varies from one ordered lead to the other through a disordered segment. This will subtend a closed circuit in the two-dimensional parameter space $R(U(x), \eta(x))$. The question we are addressing now is *whether or not there is a geometric phase associated with the circuit and its magnitude if there is any.* Such an anholonomy is expected from the form of the evolution equation (5) even though the evolution is non-unitary. The essential feature is the non-commutativity of the diagonal and the off-diagonal parts of the Hamiltonian H_{eff}. *It is important to note here that the circuit in the parameter space depends explicitly on the energy E.* We take the random potentials to vary slowly in the x-space so as to make the evolution adiabatic in the sense of the geometric phase.

As noted earlier, a first-principle calculation of the geometric phase is given in ref. [2] for an adiabatic and unitary evolution of the Hamiltonian on a closed circuit in a parameter space. Calculation of the geometric phase was later generalized for the non-unitary evolution [5, 6]. The geometric phase γ for an adiabatic evolution in terms of the instantaneous right-eigenstate is given by

$$\gamma_{\pm} = i \oint \frac{\langle \pm | \partial \vec{R} \pm \rangle}{\langle \pm | \pm \rangle} \mathrm{d}\vec{R}, \tag{8}$$

where $\vec{R}$ traces the circuit in the two-dimensional parameter spaces $R((E - U), \eta)$. Now, from

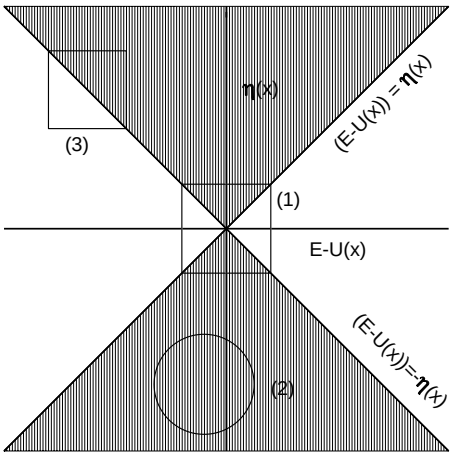

Fig. 1. – Parameter space $R((E-U(x)),\eta(x))$ for the adiabatic circuit. The shaded portion marks the regime where geometric phase shift exists, that is $|(E-U(x))| < |\eta(x)|$.

eq. (8), the values of the geometric phases for the instantaneous eigenvectors $|+\rangle$ and $|-\rangle$ are

$$\gamma_{\pm} = \mp \oint \frac{1}{2}\left[\frac{1/\eta\ d(E-U)}{\sqrt{1-(E-U)^2/\eta^2}} - \frac{(E-U)/\eta^2\ d\eta}{\sqrt{1-(E-U)^2/\eta^2}}\right]. \tag{9}$$

From eq. (8), it is clear that in order to have a non-zero geometric phase, the "instantaneous" eigenvector has to be complex. The eigenvector expressions in eq. (7) give $|\pm\rangle$ to be complex if $|\eta(x)| > |E-U(x)|$. In fig. 1 we have drawn the parameter space $R((E-U(x)),\eta(x))$ with typical circuits marked (1), (2) or (3). The region $|\eta(x)| > |E-U(x)|$ is shown in shade, where eigenvectors are complex and the geometric phase shift arises. In the non-shaded region the eigenvectors are real and have no geometric phases associated with them. Straightforward application of Stokes theorem shows that the total geometric phase associated with a closed circuit in the parameter space is zero except for the case of circuits encircling the singular point $E-U=0$, $\eta=0$. Also, in the latter case, the phase shift is independent of the particular circuit traversed. We have, therefore, chosen a simple circuit marked (1) as shown in fig. 1 for computational case. We get (setting $(E-U)/\eta = z$)

$$\gamma_{\pm} = \mp\frac{1}{2}\left[2\sin^{-1} z|_{-1}^{+1}\right] = \pm\pi \pmod{2\pi}. \tag{10}$$

This is our main result.

It will be apt at this stage to discuss the validity of eq. (10). In particular, we would like to make sure that the approximations of the continuum limit, of adiabaticity and the consideration of localization do not drastically limit the domain of validity of eq. (10). First, the continuum limit (eq. (2)) in relation to the singular points where the right-hand side of eq. (6) vanishes, giving the topological phase jump. In the absence of disorder, the dimer-to-dimer spatial variation of the two-component wave function for the discrete dimerized lattice is controlled by the wave vector, k, for a given energy eigenvalue, E. With disorder, one has to consider the local wave vector $k(x)$, which vanishes at the "turning points" $E-U=0=\eta$, where the local energy eigenvalue $\lambda_{\pm}$ vanishes. This ensures a slow spatial variation as $k(x)s=0$ at these points, where s is the lattice spacing. This justifies our continuum limit at the point $E-U(x)=0=\eta(x)$, that generates the topological phase shift. The topological

nature of this phase shift then ensures validity of our treatment in the neighbourhood of this singular point. The point to note here is that the slowness of variation of the wave function, relevant to our continuum limit, is over dimer-to-dimer spacing, and *not intra-dimer*. The latter is *absorbed* in the two-component nature of our wave function. This is quite analogous to the two-component Dirac continuum limit of the two-band models, treated in the literature [7]. In our case, the Dirac equation is massless. Next, we consider the effect of localization. First, one can see from eq. (3) that the disorder $U(x)$ can give only forward scattering and hence cannot be very effective for localization. It is the disorder $\eta(x)$ that gives back-scattering and is effective for localization. However, both $U(x)$ and $\eta(x)$ control the path in the parameter space, and, therefore, whether or not the path encircles the singular point. One can thus have weak disorder, so as to have the localization length exceed the sample length and still not invalidate the continuum limit in view of our discussion above. Slow variation of disorder in space can ensure adiabaticity. Thus, in principle, the parameter space for the validity of our treatment is not constrained by conditions for continuum limit, localization and adiabaticity, militating against one another.

The question now is what observable consequences this geometric (topological) phase may have. Given that it is the relative, rather than the absolute, phase that has a physical significance, one has basically two ways of demonstrating the geometric-phase sensitive effects. Either prepare the quantum system in a coherent superposition of two instantaneous eigenstates that pick up different geometric phases for the same path traversed in the parameter space (*i.e.*, two states but one Hamiltonian), or prepare the system in a given initial (instantaneous) eigenstate and let it evolve along two different alternative paths traversing the parameter spaces so as to give an interference between the partial amplitudes (one state and two Hamiltonians). The physical significance of the above geometric (topological) phase is, however, best realized by considering the following situation. Consider a 1D disordered dimerized conductor in the form of a ring. Such a system automatically ensures a closed circuit in the parameter space as discussed above. Mathematically, this means imposing the periodic boundary condition. The periodic boundary condition involves the matching of the phases —the phase change accumulated around the ring must be a multiple of 2π. But now this must include the extra geometric (topological) phase as well. This will alter the eigenvalue spectrum for the ring and must be included in any spectral reckoning. In the limit of weak scattering, *i.e.* disorder parameter $\ll$ level spacing (which can be obtained for small enough ring size, the mesoscopic system) this extra phase of $\pm\,\pi$ will cause a level shift. A rather subtle and experimentally observable effect associated with this geometric phase is made plausible if we consider the disorder $U(x)$ and $\eta(x)$ to be weakly and slowly modulated in time. This has the effect of translating the parametric (sample specific) fluctuation into temporal fluctuations. This happens naturally in mesoscopic systems at low temperatures. The modulation depth can be such that the circuit in the $U(x)$ and $\eta(x)$ parameter space sweeps across the central singularity ($\eta(x) = 0 = E - U(x)$) —*i.e.* the geometric phase is cyclically flipped from zero to $\pm\,\pi$ *discontinuously*. The discontinuous nature of our topological phase change should lead to a *large spectral (level spacing) and conductance fluctuations manifesting as noise*. The discreteness of the topological phase shift (as distinct from the usual continuous geometric phase) should make this the statistical analogue of a shot noise. Finally, we must note that the circuit in the $U(x)$ and $\eta(x)$ parameter space involves energy E explicitly. Thus, for a given realization of randomness, as E varies we expect singularities in the energy spectrum at the special values of E corresponding to the circuit enclosing, or not enclosing, the center, *i.e.*, the singularity of the parameter space. This suggests the following experimentally realizable possibility of observing certain phase-sensitive effects by combining our pseudo-magnetic field with a real Aharonov-Bohm magnetic flux, threading the matrial ring connected to two leads.

The variation of the energy level as a function of the real flux can then tune it across the singular point causing the phase jump. This must manifest as jump in the transmission through the ring. Similarly, one should expect jumps in the persistent ring currents.

We would like to conclude with the following remarks in support of what is new in our work. Topological phase shift is by itself not a new idea now. It is a well-known particular case of Berry's phase when, *e.g.*, the magnetic field acting on a spin is confined to a planar transport circuit. Indeed, such situations have been realised literally experimentally involving, *e.g.*, a momentum-dependent magnetic field via the spin-orbit coupling [8]. In our work, the pseudo-magnetic field arises in a novel way through the dimerised nature of the medium, and one has to calculate the topological phase shift for the effectively non-unitary evolution of a properly defined state vector. Further, the parametric trajectory traced out by this pseudo-magnetic field, as one goes along the sample length, depends sensitively on the sample specific realization of the disorder. It is this sensitivity coupled with the discreteness of the topological phase shift that generates the shot noise when the sample scans the parameter space through, *e.g.*, a phononic modulation. Clearly, other manifestations of this topological phase are expected, *e.g.*, through its effect on the level spacing statistics. We hope that this work will stimulate experimental work on mesoscopic rings made from quasi–one-dimensional disordered dimerized materials, *i.e.* materials having bipartite lattice structure. Many conducting pseudo–one-dimensional polymeric systems may belong to this category of materials.

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