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Continuously guided atomic interferometry using a single-zone optical excitation: theoretical analysis

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Abstract

In an atomic interferometer, the phase shift due to rotation is proportional to the area enclosed by the split components of the atom. However, this model is unclear for an atomic interferometer demonstrated recently by Shahriar et al., for which the atom simply passes through a *single-zone* optical beam, consisting of a pair of bichromatic counter-propagating beams. During the passage, the atomic wave packets in two distinct internal states couple to each other *continuously*. The two internal states trace out a complicated trajectory, guided by the optical beams, with the amplitude and spread of each wavepacket varying continuously. Yet, at the end of the single-zone excitation, there is an interference with fringe amplitudes that can reach a visibility close to unity. For such a situation, it is not clear how one would define the area of the interferometer, and therefore, what the rotation sensitivity of such an interferometer would be. In this paper we analyze this interferometer in order to determine its rotation sensitivity, and thereby determine its effective area. In many ways, the continuous interferometer (CI) can be thought of as a limiting version of the Borde–Chu Interferometer (BCI). We identify a quality factor that can be used to compare the performance of these interferometers. Under conditions of practical interest, we show that the rotation sensitivity of the CI can be comparable to that of the BCI. The relative simplicity of the CI (e.g., elimination of the task of precise angular alignment of the three zones) then makes it a potentially better candidate for practical atom interferometry for rotation sensing.

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1. Introduction

In an atomic interferometer[1–8], the phase shift due to rotation is proportional to the area enclosed by the split components of the atom. In most situations, the atomic wavepacket is split first by what can be considered effectively as an atomic beam-splitter [9–13]. The split components are then redirected towards each other by atomic mirrors. Finally, the converging components are recombined by another atomic beam splitter. Under these conditions, it is simple to define the area of the interferometer by considering the center of mass motion of the split components. However, this model is invalid for an atomic interferometer demonstrated recently by Shahriar et al. [14]. Briefly, in this interferometer, the atom simply passes through a *single-zone* optical beam, consisting of a pair of bichromatic counter-propagating beams. During the passage, the atomic wave packets in two distinct internal states couple to each other *continuously*. The two internal states trace out complicated trajectories, guided by the optical beams, with the amplitude and spread of each wavepacket varying continuously. Yet, at the end of the single-zone excitation, there is an interference with fringe amplitudes that can reach a visibility close to unity. For such a situation, it is not clear how one would define the area of the interferometer, and therefore, what the rotation sensitivity of such an interferometer would be.

In this paper we analyze this interferometer in order to determine its rotation sensitivity, and thereby determine its effective area. In many ways, the continuous interferometer (CI) can be thought of as a limiting version of the three-zone interferometer proposed originally by Borde [1], and demonstrated by Kasevich and Chu [2]. In our analysis, we compare the behavior of the CI with the Borde–Chu Interferometer (BCI). We also identify a quality factor that can be used to compare the performance of these interferometers. Under conditions of practical interest, we show that the rotation sensitivity of the CI can be comparable to that of the BCI. The relative simplicity of the CI (e.g., the task of precise angular alignment of the three zones is eliminated for the CI) then makes it a potentially better candidate for practical atom interferometry for rotation sensing.

In our comparative analysis, we find it more convenient to generalize the BCI by making the position and duration of the phase-scanner a variable. As such, we end up comparing two types of atomic interferometers to the BCI. The first, which is the generalized version of the BCI, is where instead of a phase scan being applied in only the final $\pi/2$ pulse, the phase scan is applied from some point onwards in the middle π pulse. We find that the magnitude of the rotational phase shift varies according to where the phase is applied from. This phase shift is calculated analytically and compared to the phase shift obtained in the original BCI. The second type of interferometer that we compare to the original BCI is the CI, where the atom propagates through only one laser beam that has a Gaussian field profile. The atom is modeled as a wavepacket with a Gaussian distribution in the momentum representation, and it's evolution in the laser field is calculated numerically. From this, the rotational phase shift is obtained and compared once again to the phase shift in the BCI.

2. Formulation of the problem

We model the system as a three level atom in the lambda configuration, as shown in Fig. 1, with levels $|a\rangle$, $|b\rangle$, and $|e\rangle$, which moves in the **x**-direction through two counter propagating laser beams. The laser beams travel in the **z**-direction and have Gaussian electric field profiles varying in the **x**-direction. In the electric dipole approximation, which is valid for our system since the wavelength of the light is much greater than the separation between the electron and the nucleus, we can write the interaction Hamiltonian as $\mathbf{r} \cdot \mathbf{E}$, where $\mathbf{r}$ is the position of the electron and $\mathbf{E}$ is the electric field of the laser. The states of the three level atom are driven by the laser fields. The fields cause transitions between the states $|a\rangle$ and $|e\rangle$ and the states $|e\rangle$ and

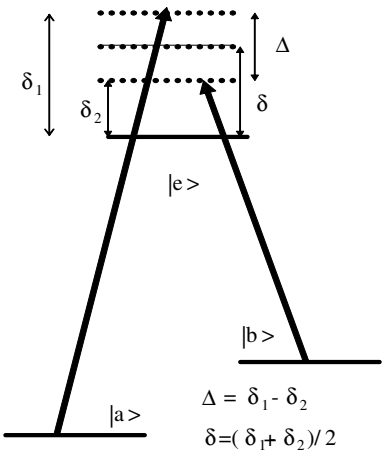

Fig. 1. A schematic picture of a three level system where δ is the common detuning and Δ is the difference detuning.

$|b\rangle$. In our analysis, we quantize the center-of-mass (COM) position of the atom in the **z**-direction. The Hamiltonian for the system can be written in the following way:

$$H = \frac{\mathbf{P}_z^2}{2m} + H_0 + \mathbf{r}\cdot\mathbf{E}_1 + \mathbf{r}\cdot\mathbf{E}_2, \tag{1}$$

where $\mathbf{E}_1$ and $\mathbf{E}_2$ are the electric field vectors of the two counter propagating lasers, $\mathbf{P}_z$ is the COM momentum in the **z**-direction, and H_0 is the internal energy. The lasers are taken to be classical electromagnetic fields. We can expand the Hamiltonian as well as the wavefunction of the COM in the basis of the eigenstates of the non-interacting Hamiltonian, which is simply: $|P_Z\rangle\otimes|i\rangle = |P_Z,i\rangle$. This is a complete set of basis states for our system. Since it is understood that all momenta and positions refer to the **z**-direction, we will drop the z subscript on all momenta from hereon.

The position operator of the electron in the atom can be expanded in terms of this basis by inserting the identity operator twice in the form

$$\hat{I} = \int \mathrm{d}p \sum_i |p,i\rangle\langle p,i|. \tag{2}$$

We also make the assumption that matrix elements of the form $\langle i|\mathbf{r}|i\rangle = 0$. Thus, in terms of the dipole matrix elements $\boldsymbol{d}_{ij} = \langle i|\boldsymbol{r}|j\rangle$, we can write the position operator as:

$$\begin{aligned}\mathbf{r} &= \int \mathrm{d}p \int \mathrm{d}p' \sum_{i,j} |p,i\rangle\langle p,i|\mathbf{r}|p',j\rangle\langle p',j| \\ &= \int \mathrm{d}p(\mathbf{d}_{\mathrm{ae}}|p,a\rangle\langle p,e| + \mathbf{d}_{\mathrm{ea}}|p,e\rangle\langle p,a| + \mathbf{d}_{\mathrm{be}}|p,b\rangle\langle p,e| + \mathbf{d}_{\mathrm{eb}}|p,e\rangle\langle p,b|).\end{aligned} \tag{3}$$

Define the atomic raising and lowering operators as:

$$\boldsymbol{\sigma}_{ij} = |p,i\rangle\langle p,j|. \tag{4}$$

In terms of these operators, the position operator is

$$\mathbf{r} = \int \mathrm{d}p(\mathbf{d}_{\mathrm{ae}}\boldsymbol{\sigma}_{\mathrm{ae}} + \mathbf{d}_{\mathrm{ea}}\boldsymbol{\sigma}_{\mathrm{ea}} + \mathbf{d}_{\mathrm{be}}\boldsymbol{\sigma}_{\mathrm{be}} + \mathbf{d}_{\mathrm{eb}}\boldsymbol{\sigma}_{\mathrm{eb}}). \tag{5}$$

Since the electric field is being modeled classically, we can express each laser field as

$$\mathbf{E}(z,t) = \mathbf{E}_0 \cos(\omega t - k\,\widehat{z} + \phi) = \frac{\mathbf{E}_0}{2}\Big(\exp(\mathrm{i}(\omega t - k\,\widehat{z} + \phi)) + \exp(-\mathrm{i}(\omega t - k\,\widehat{z} + \phi))\Big), \tag{6}$$

where $\widehat{z}$ is the operator associated with the COM position of the atom. The first laser interacts only with that part of the electron position operator which causes transitions $|a\rangle \leftrightarrow |e\rangle$, and the second laser interacts with the part which causes transitions $|b\rangle \leftrightarrow |e\rangle$. We also assume that the dipole matrix elements are real: $\mathbf{d}_{\mathrm{ae}} = \mathbf{d}_{\mathrm{ea}} = \mathbf{d}^*_{\mathrm{ae}}$. Therefore,

$$\begin{aligned}\mathbf{r}\cdot\mathbf{E}_1 = \int \mathrm{d}p \frac{\mathbf{d}_{\mathrm{ae}}\cdot\mathbf{E}_{01}}{2}\Big[&\sigma_{\mathrm{ae}}\exp\Big(\mathrm{i}\Big(\omega_1 t - k_1\,\widehat{z} + \phi_1\Big)\Big) + \sigma_{\mathrm{ae}}\exp\Big(-\mathrm{i}\Big(\omega_1 t - k_1\,\widehat{z} + \phi_1\Big)\Big)\\ &+ \sigma_{\mathrm{ea}}\exp\Big(\mathrm{i}\Big(\omega_1 t - k_1\,\widehat{z} + \phi_1\Big)\Big) + \sigma_{\mathrm{ea}}\exp\Big(-\mathrm{i}\Big(\omega_1 t - k_1\,\widehat{z} + \phi_1\Big)\Big)\Big].\end{aligned} \tag{7}$$

Now we make the standard rotating wave approximation which neglects the terms in this expression which do not conserve energy. Also, let $\Omega_1 = (\boldsymbol{d}_{\mathrm{ae}}\cdot\boldsymbol{E}_{01})/\hbar$, so that

$$\mathbf{r}\cdot\mathbf{E}_1 = \int \mathrm{d}p \frac{\hbar\Omega_1}{2}\Big[\sigma_{\mathrm{ae}}\exp\Big(\mathrm{i}\Big(\omega_1 t - k_1\,\widehat{z} + \phi_1\Big)\Big) + \sigma_{\mathrm{ea}}\exp\Big(-\mathrm{i}\Big(\omega_1 t - k_1\,\widehat{z} + \phi_1\Big)\Big)\Big], \tag{8}$$

with a corresponding expression for the $\mathbf{r}\cdot\mathbf{E_2}$ part. Now, we get:

$$\begin{aligned}\exp(\mathrm{i}k\hat{z}) &= \sum_{i,j}\int \mathrm{d}p \int \mathrm{d}p' |p,i\rangle\langle p,i|\mathrm{e}^{\mathrm{i}kz}|p',j\rangle\langle p',j|\\ &= \sum_{i,j}\int \mathrm{d}p \int \mathrm{d}p' \int \mathrm{d}z \int \mathrm{d}z' |p,i\rangle\langle p,i \mid z,i\rangle\langle z,i|\mathrm{e}^{\mathrm{i}kz}|z',j\rangle\langle z',j \mid p',j\rangle\langle p',j|\\ &= \sum_{i,j}\int \mathrm{d}p \int \mathrm{d}p' \int \mathrm{d}z \int \mathrm{d}z' |p,i\rangle \mathrm{e}^{\frac{-ipz}{\hbar}}\mathrm{e}^{\mathrm{i}kz'}\mathrm{e}^{\frac{ip'z'}{\hbar}}\delta_{ij}\delta(z-z')\langle p',j|\\ &= \sum_{i}\int \mathrm{d}p \int \mathrm{d}p' |p,i\rangle\langle p',j|\delta\left(k - \frac{p}{\hbar} + \frac{p'}{\hbar}\right)\\ &= \sum_{i}\int \mathrm{d}p |p,i\rangle\langle p-\hbar k,i|\end{aligned} \tag{9}$$

and

$$\exp(-\mathrm{i}k\hat{z}) = \sum_i \int \mathrm{d}p |p,i\rangle\langle p+\hbar k,i|. \tag{10}$$

The expansion of the non-interacting part of the Hamiltonian in Eq. (1) in terms of this basis is

$$H_0 = \int \mathrm{d}p \sum_i \left(\frac{p^2}{2m} + \hbar\omega_i\right)\sigma_{ii}, \tag{11}$$

where $\hbar\omega_i$ is the energy of the ith level.

Combining expressions of Eqs. (7)–(11) in Eq. (1), we finally get the full Hamiltonian in the $|p,i\rangle$ basis:

$$\begin{aligned}H = \int \mathrm{d}p \Bigg[&\sum_i \left(\frac{p^2}{2m} + \hbar\omega_i\right)\sigma_{ii} + \frac{\hbar\Omega_1}{2}\left(|p,a\rangle\langle p+\hbar k_1,e|\mathrm{e}^{\mathrm{i}(\omega_1 t+\phi_1)} + |p+\hbar k_1,e\rangle\langle p,a|\mathrm{e}^{-\mathrm{i}(\omega_1 t+\phi_1)}\right)\\ &+ \frac{\hbar\Omega_2}{2}\left(|p,b\rangle\langle p+\hbar k_2,e|\mathrm{e}^{\mathrm{i}(\omega_2 t+\phi_2)} + |p+\hbar k_2,e\rangle\langle p,b|\mathrm{e}^{-\mathrm{i}(\omega_2 t+\phi_2)}\right)\Bigg].\end{aligned} \tag{12}$$

For a given value of the momentum p, it is clear that this Hamiltonian creates transitions only between the following manifold of states, $|p,a\rangle \leftrightarrow |p+\hbar k_1,e\rangle \leftrightarrow |p+\hbar k_1-\hbar k_2,b\rangle$. That is, the only way to transition between states $|a\rangle$ and $|b\rangle$ is to pass through $|e\rangle$ and make the accompanying momentum transitions as indicated. Therefore it is convenient to make the following substitutions of the momentum variables:

$$\int \mathrm{d}p\left(\frac{p^2}{2m}+\hbar\omega_e\right)|p,e\rangle\langle p,e| = \int \mathrm{d}q_1\left(\frac{(q_1+\hbar k_1)^2}{2m}+\hbar\omega_e\right)|q_1+\hbar k_1,e\rangle\langle q_1+\hbar k_1,e|, \tag{13a}$$

$$\int \mathrm{d}p\left(\frac{p^2}{2m}+\hbar\omega_b\right)|p,b\rangle\langle p,b| = \int \mathrm{d}q_2\left(\frac{(q_2+\hbar k_1-\hbar k_2)^2}{2m}+\hbar\omega_b\right) \times |q_2+\hbar k_1-\hbar k_2,b\rangle\langle q_2+\hbar k_1-\hbar k_2,b|, \tag{13b}$$

$$\int \mathrm{d}p|p,b\rangle\langle p+\hbar k_2,e| = \int \mathrm{d}q_2|q_2+\hbar k_1-\hbar k_2,b\rangle\langle q_2+\hbar k_1,e|. \tag{13c}$$

Thus if we define the states

$$|1\rangle = |p,a\rangle, \tag{14a}$$

$$|2\rangle = |p+\hbar k_1,e\rangle, \tag{14b}$$

$$|3\rangle = |p+\hbar k_1-\hbar k_2,b\rangle, \tag{14c}$$

we can rewrite the Hamiltonian (Eq. (12)) as

$$H = \int \mathrm{d}p\left[\sum_i \varepsilon_i\sigma_{ii} + \frac{\hbar\Omega_1}{2}\left(|1\rangle\langle 2|\mathrm{e}^{\mathrm{i}(\omega_1 t+\phi_1)} + |2\rangle\langle 1|\mathrm{e}^{-\mathrm{i}(\omega_1 t+\phi_1)}\right) + \frac{\hbar\Omega_2}{2}\left(|3\rangle\langle 2|\mathrm{e}^{\mathrm{i}(\omega_2 t+\phi_2)} + |2\rangle\langle 3|\mathrm{e}^{-\mathrm{i}(\omega_2 t+\phi_2)}\right)\right], \tag{15}$$

where the ε_i are the energies of the newly defined states. This form of the Hamiltonian makes it clear that once the atom has some momentum p, the interaction cannot move it to a manifold of states with some other momentum. The only transitions that can occur are between the states $|1\rangle$, $|2\rangle$ and $|3\rangle$, for the given momentum. Thus, to study the dynamics of the atom, it is sufficient to consider only one manifold with some momentum p. Once solved, we can integrate over all momenta to get the motion of the full wavepacket.

Since the laser beams are counter-propagating at the same frequency, we have $k_1 = -k_2 = k$, and the states become:

$$|1\rangle = |p,a\rangle, \tag{16a}$$

$$|2\rangle = |p+\hbar k,e\rangle, \tag{16b}$$

$$|3\rangle = |p+2\hbar k,b\rangle. \tag{16c}$$

The state of the atom is expanded in the $|p,i\rangle$ basis as

$$\begin{aligned}|\Psi(t)\rangle &= \int \mathrm{d}p\sum_i \psi_i(p,t)|p,i\rangle \\ &= \int \mathrm{d}p(\alpha(p,t)|p,a\rangle + \beta(p+2\hbar k,t)|p+2\hbar k,b\rangle + \xi(p+\hbar k,t)|p+\hbar k,e\rangle),\end{aligned} \tag{17}$$

and evolves according to the Schreodinger equation:

$$i\hbar\frac{\partial|\Psi\rangle}{\partial t} = H\Psi. \tag{18}$$

If we make a unitary transformation U on the state $|\Psi\rangle$ to some interaction-picture state vector $|\tilde{\Psi}\rangle = U|\Psi\rangle$, then the Hamiltonian in this interaction picture is

$$\tilde{H} = UHU^{-1} + i\hbar\frac{\partial U}{\partial t}U^{-1}. \tag{19}$$

Let $U = \int dp \sum_j e^{i(\theta_j t+\varsigma_j)}|j\rangle$, where the $|j\rangle$ are the redefined states of Eq. (16a)–(16c), and the θ_j and ς_j are parameters we will choose to simplify the interaction picture Hamiltonian. Written in matrix form, the Hamiltonian for some momentum p is

$$H(p) = \begin{bmatrix} \varepsilon_1 & 0 & \frac{\hbar\Omega_1}{2}e^{i(\omega_1 t+\phi_1)} \\ 0 & \varepsilon_3 & \frac{\hbar\Omega_2}{2}e^{i(\omega_2 t+\phi_2)} \\ \frac{\hbar\Omega_1}{2}e^{-i(\omega_1 t+\phi_1)} & \frac{\hbar\Omega_2}{2}e^{-i(\omega_2 t+\phi_2)} & \varepsilon_2 \end{bmatrix}, \tag{20}$$

where the rows and columns are arranged with the states in the $\{|1\rangle, |3\rangle, |2\rangle\}$ order. In the interaction picture with the parameters θ_j and ς_j, the Hamiltonian is

$$\tilde{H}(p) = \begin{bmatrix} \varepsilon_1 - \hbar\theta_1 & 0 & \frac{\hbar\Omega_1}{2}e^{i(\omega_1+\theta_1-\theta_2)t+i(\phi_1+\varsigma_1-\varsigma_2)} \\ 0 & \varepsilon_3 - \hbar\theta_3 & \frac{\hbar\Omega_2}{2}e^{i(\omega_2+\theta_3-\theta_2)t+i(\phi_2+\varsigma_3-\varsigma_2)} \\ \frac{\hbar\Omega_1}{2}e^{-i(\omega_1+\theta_1-\theta_2)t-i(\phi_1+\varsigma_1-\varsigma_2)} & \frac{\hbar\Omega_2}{2}e^{-i(\omega_2+\theta_3-\theta_2)t-i(\phi_2+\varsigma_3-\varsigma_2)} & \varepsilon_2 - \hbar\theta_2 \end{bmatrix}. \tag{21}$$

First, to get rid of the time dependence, set $\omega_1 + \theta_1 - \theta_2 = 0$, and $\omega_2 + \theta_3 - \theta_2 = 0$. Also, set $\varsigma_1 = -\phi_1$, $\varsigma_2 = 0$, and $\varsigma_3 = -\phi_2$. Define the detunings:

$$\hbar\delta_1 = \varepsilon_1 + \hbar\omega_1 - \varepsilon_2, \tag{22a}$$

$$\hbar\delta_2 = \varepsilon_3 + \hbar\omega_2 - \varepsilon_2, \tag{22b}$$

$$\hbar\Delta = \hbar(\delta_1 - \delta_2), \tag{22c}$$

$$\hbar\delta = \frac{\hbar(\delta_1 + \delta_2)}{2}. \tag{22d}$$

A consistent choice of the θ parameters that also simplifies the form of the Hamiltonian considerably is:

$$\hbar\theta_1 = \frac{\varepsilon_1 + \varepsilon_3 - \hbar\omega_1 + \hbar\omega_2}{2}, \tag{23a}$$

$$\hbar\theta_2 = \frac{\varepsilon_1 + \varepsilon_3 + \hbar\omega_1 + \hbar\omega_2}{2}, \tag{23b}$$

$$\hbar\theta_3 = \frac{\varepsilon_1 + \varepsilon_3 + \hbar\omega_1 - \hbar\omega_2}{2}. \tag{23c}$$

With this choice, the Hamiltonian becomes

$$\tilde{H}(p) = \hbar\begin{bmatrix} \frac{\Delta}{2} & 0 & \frac{\Omega_1}{2} \\ 0 & -\frac{\Delta}{2} & \frac{\Omega_2}{2} \\ \frac{\Omega_1}{2} & \frac{\Omega_1}{2} & -\delta \end{bmatrix} \tag{24}$$

which is familiar from semi-classical (i.e., without quantization of the COM motion) descriptions of the three-level interaction [15–19], keeping in mind that here it represents the Hamiltonian only within a given manifold. The equations of motion for these three states within the manifold of a given momentum are:

$$i\hbar\dot{\tilde{\alpha}}(p,t) = \frac{\hbar\Delta}{2}\tilde{\alpha}(p,t) + \frac{\hbar\Omega_1}{2}\tilde{\xi}(p+\hbar k,t), \tag{25a}$$

$$i\hbar\dot{\tilde{\beta}}(p+2\hbar k,t) = -\frac{\hbar\Delta}{2}\tilde{\beta}(p+2\hbar k,t) + \frac{\hbar\Omega_2}{2}\tilde{\xi}(p+\hbar k,t), \tag{25b}$$

$$i\hbar\dot{\tilde{\xi}}(p+\hbar k,t) = -\hbar\delta\tilde{\xi}(p+\hbar k,t) + \frac{\hbar\Omega_1}{2}\tilde{\alpha}(p,t) + \frac{\hbar\Omega_2}{2}\tilde{\beta}(p+2\hbar k,t). \tag{25c}$$

Since the laser beams are far detuned from resonances, we make the adiabatic approximation, which can be verified afterwards for consistency. This approximation assumes that the intermediate $|2\rangle$ state occupation is negligible and that we can set $\dot{\tilde{\xi}} \approx 0$. This allows us to reduce this three level system to a two level system by solving for $\tilde{\xi}$ in Eq. (25c), and then substituting it into Eqs. (25a) and (25b). We get

$$\tilde{\xi} = \frac{\Omega_1}{2\delta}\tilde{\alpha} + \frac{\Omega_2}{2\delta}\tilde{\beta}, \tag{26}$$

and the effective two level Hamiltonian

$$\tilde{H}_{\text{eff}}(p) = \hbar\begin{bmatrix} \frac{\Delta}{2} + \frac{\Omega_1^2}{4\delta} & \frac{\Omega_1\Omega_2}{4\delta} \\ \frac{\Omega_1\Omega_2}{4\delta} & -\frac{\Delta}{2} + \frac{\Omega_2^2}{4\delta} \end{bmatrix}. \tag{27}$$

In our system, we assume that the counter-propagating laser beams have the same strength, $\Omega_1 = \Omega_2$, and define the effective Raman Rabi frequency $\Omega_0 = \frac{\Omega_1\Omega_2}{2\delta}$. Thus the effective Hamiltonian becomes

$$\tilde{H}_{\text{eff}}(p) = \hbar\begin{bmatrix} \frac{\Delta}{2} + \frac{\Omega_0}{2} & \frac{\Omega_0}{2} \\ \frac{\Omega_0}{2} & -\frac{\Delta}{2} + \frac{\Omega_0}{2} \end{bmatrix}. \tag{28}$$

The expressions for the detunings are:

$$\Delta = \Delta_0 - \frac{2kp}{m} - \frac{2\hbar k^2}{m}, \tag{29a}$$

$$\delta = \delta_0 + \frac{\hbar k^2}{2m}, \tag{29b}$$

where $\Delta_0 = \omega_1 - \omega_2 + \omega_a - \omega_b$ and $\delta_0 = (\omega_1 + \omega_2 + \omega_a + \omega_b - 2\omega_e)/2$. This effective Hamiltonian can be solved by standard methods for $\tilde{\alpha}(p,t)$ and $\tilde{\beta}(p+2\hbar k,t)$. Once we have the solutions for some Ω_0 and at a given value of p, then we can write down the full expression for the state vector integrated over all p. Ignoring any global phase factors which do not depend on p, we get

$$\begin{aligned} |\Psi(t)\rangle &= \int \mathrm{d}p(\alpha(p,t)|p,a\rangle + \beta(p+2\hbar k,t)|p+2\hbar k,b\rangle) \\ &= \int \mathrm{d}p\left(\mathrm{e}^{-i\theta_1 t}\tilde{\alpha}(p,t)|p,a\rangle + \mathrm{e}^{-i\theta_3 t}\tilde{\beta}(p+2\hbar k,t)|p+2\hbar k,b\rangle\right) \\ &= \int \mathrm{d}p \exp\left(\frac{-i\left(p^2 + (p+2\hbar k)^2\right)}{4m\hbar}t\right)\left(\tilde{\alpha}(p,t)|p,a\rangle + \tilde{\beta}(p+2\hbar k,t)|p+2\hbar k,b\rangle\right). \end{aligned} \tag{30}$$

In our analysis of the rotational sensitivity, we must apply this solution for the state vector for the case of a Gaussian profile in the x-direction. We simply discretize the Gaussian profile and propagate stepwise along the discrete profile until we reach the time desired. The position representation of the wavefunctions for the $|a\rangle$ and $|b\rangle$ states are then:

$$\psi_a(z,t) = \int \mathrm{d}p\alpha(p,t)\exp\left(\frac{-\mathrm{i}pz}{\hbar}\right), \tag{31a}$$

$$\psi_b(z,t) = \int \mathrm{d}p\beta(p+2\hbar k,t)\exp\left(\frac{-\mathrm{i}pz}{\hbar}\right), \tag{31b}$$

and the probabilities for the atom to be in either state are:

$$P(a) = \int \mathrm{d}p|\alpha(p,t)|^2, \tag{32a}$$

$$P(b) = \int \mathrm{d}p|\beta(p,t)|^2. \tag{32b}$$

3. Rotational sensitivity

In the setup for the Borde–Chu Interferometer, shown in Fig. 2, where we assume that the transverse displacement is negligible as compared to the longitudinal travel ($L \gg d$), the phase shift due to rotation of the interferometer may be interpreted as resulting from the deviation in position of the laser beams with respect to the atomic trajectories. This can be seen as follows. When the BCI is stationary, the laser fields may be assumed, without loss of generality, not to have any phase difference relative to one another. Once the BCI begins rotating with some angular velocity Ω around an arbitrary axis, each of the laser beams will move a distance relative to the axis of rotation in proportion to Ω. This deviation in position results in a phase shift in each of the laser fields, given by $2k\Delta y$, where k is the wave number of the lasers and Δy is the change in position. The factor of 2 results from the fact that two counter-propagating beams are used in each zone. The total phase shift due to the lasers in the BCI is $\delta\phi = \phi_1 - 2\phi_2 + \phi_3$, where ϕ_i ($i = 1,2,3$) corresponds to the phase deviation of the ith laser field. We assume, without loss of generality, that $\phi_i = 0$ ($i = 1,2,3$) in the absence of any rotation, or any external phase shift applied to the beams. The rotational phase $\delta\phi_0$ is calculated by taking into account the phase shift of each laser field resulting from its respective

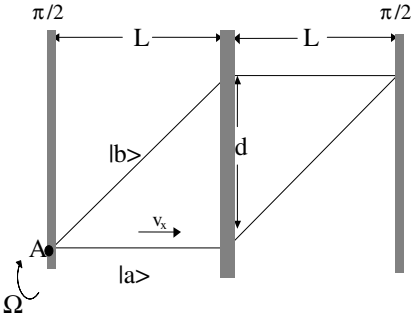

Fig. 2. Schematic of the setup for the Borde–Chu Interferometer. The atom moves at some speed v_x, with the two atomic states deflected along different paths. The entire system is rotating around point A at some rotational velocity Ω.

change in position, by the time the atom reaches that field. If we choose the interferometer to rotate around point A, for example, then $\Delta y_1 = 0$, $\Delta y_2 = L\Omega T$, $\Delta y_3 = 4L\Omega T$, where T is the time the atom takes to go between lasers. Thus the phases associate with the rotation of the interferometer: $\phi_{10} = 0$, $\phi_{20} = 2kL\Omega T$, $\phi_{30} = 8kL\Omega T$, and $\delta\phi_0 = 4kL\Omega T$. Since $v_y = 2\hbar k/m$ (from photon recoil), $d = v_y T$, and the area of the interferometer is $A_0 = Ld = 2LT\hbar$ k/m, we get

$$\delta\phi_0 = 4kL\Omega T = 2\Omega A_0 m/\hbar. \tag{33}$$

This expression for the rotational phase remains the same regardless of the position of the axis of rotation, as can be shown explicitly. We are neglecting second order contributions to the rotational phase, which come from the difference in path lengths between the upper and lower arms while rotating. Of course, even though this expression is derived here explicitly for the BCI, it is in fact applicable to any atomic interferometer as long as the area enclosed (as defined by the semi-classical trajectories) is given by A_0 [20,21]. The expression can also be derived from the corresponding expression for the rotation sensitivity of an optical gyroscope [$4\pi\Omega A_0/\lambda c$] based on the Sagnac effect [22], by substituting mc^2 (the rest energy of the atom) for $h\nu$, the photon energy.

During a typical operation of the BCI [2–4], an additional phase shift ϕ is applied on the third zone of laser beams so that $\delta\phi = \phi + \delta\phi_0$. As the value of ϕ gets scanned, the observed population of state $|b\rangle$ varies sinusoidally. Specifically, the population depends on this phase $\delta\phi$ as follows [1,2]:

$$P = \frac{1}{2}[1 - \cos(\delta\phi)]. \tag{34}$$

If there is no rotation, (i.e., $\delta\phi_0 = 0$), then the fringe minimum occurs at $\phi = 0$. In the case of a non-zero rotation, this minimum is shifted to $\phi = -\delta\phi_0$. Measurement of this shift can therefore be used to determine the angular velocity from Eq. (33).

In order to establish a framework for interpreting the behavior of the CI, let us now consider a system where instead of doing a scan by applying a phase-shift only to the third beam (the last $\pi/2$ pulse), we apply a phase-shift partway through the middle beam (the π pulse), which is of time length τ and space length l. This modified Borde–Chu Interferometer is shown in Fig. 3. These length parameters have the relationship $l = v_x\tau$, where v_x is the velocity of the atom in the **x**-direction. The phase-shift is applied starting from a

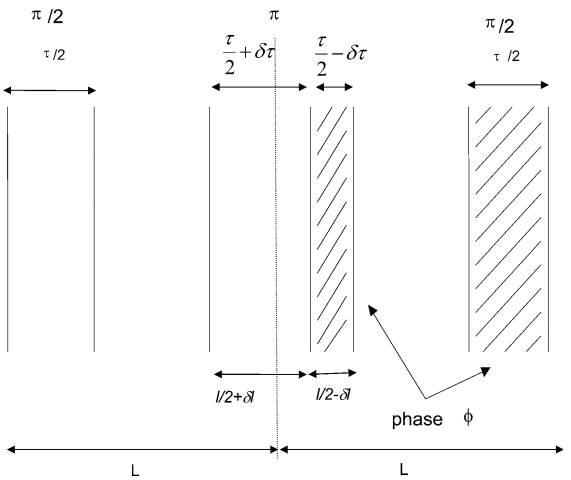

Fig. 3. Modified Borde–Chu Interferometer where the phase is applied partway through the center π pulse and through the last $\pi/2$ pulse. The phase is applied starting at δl from the middle of the π pulse.

distance δl away from the center of the π pulse. The center of the pulse corresponds to $\delta\, l = 0$, and the phase-shift ϕ is applied at all points in the beam to the right of δl. Thus, the π pulse is effectively split into two beams, the first one of length $l/2 + \delta l$ where there is no phase-shift applied, and the second of length $l/2 - \delta l$ where the phase-shift ϕ is applied.

In deriving an approximate analytic expression for the complex amplitude $c_b = \langle b|\Psi\rangle$ of the state $|b\rangle$ after passing through such an interferometer, we use the Hamiltonian given by Eq. (28) and set $\Delta = 0$. This amounts to neglecting the Doppler shift and assuming that both lasers are equally detuned. We model the π pulse as two separate beams of variable lengths. The first beam is of time length $\tau/2 + \delta\tau$, and the second is $\tau/2-\delta\tau$. Letting $\tau_2 = \tau/2-\delta\tau$ and noting that $\Omega_0\tau = \pi$, where Ω_0 is the Raman Rabi frequency, ω_b is the free space propagation frequency, we can derive the expression for the amplitude of the excited state at the end of the $\pi/2$ pulse at the 3rd zone,

$$c_b = \frac{i}{2}\exp(-i\omega_b 2T)\left[\sin\left(\frac{\Omega_0\tau_2}{2}\right)\cos\left(\frac{\Omega_0\tau_2}{2}\right)(\exp(-i\phi_{30})-1)(1-\exp(-i\phi)) + \cos^2\left(\frac{\Omega_0\tau_2}{2}\right)\right.$$
$$\left.\times\exp(-i\phi_{20})(\exp(-i\delta\phi_0 - i\phi)-1) + \sin^2\left(\frac{\Omega_0\tau_2}{2}\right)\exp(-i\phi_{20})(\exp(-i\delta\phi_0)-\exp(-i\phi))\right]. \quad (35)$$

In the limit that $\tau_2 = 0$ or $\tau_2 = \tau$, and if the rotation velocity Ω is 0, the absolute value squared of this equation reduces to Eq. (34) for the fringes. Several parts of the formalism developed above for the modified BCI could also be interpreted using the Bloch-vector formalisms [23,24].

We are now in a position to compare the rotation sensitivity of the original BCI and that of this modified configuration where the phase is applied starting at some point in the π pulse. First, we define the effective area A_{eff} for the modified BCI as a proportionality constant between the calculated fringe shift $\Delta\,\phi$ and the rotation rate Ω, in the following form:

$$\Delta\phi = -2m\Omega A_{\text{eff}}/\hbar. \quad (36)$$

This effective area may or may not be the same as the true area of the interferometer. In order to determine the value of A_{eff}, we require an expression for this fringe shift that results upon rotation for this new system. To derive this, we first take the absolute square of Eq. (35). We then take the derivative of the absolute square with respect to ϕ to find the minimum and compare how far the minimum shifts as a function of rotation. After a tedious but straightforward calculation we get an exact expression for the phase shift $\Delta\,\phi$:

$$\Delta\phi = \tan^{-1}\frac{[\sin^4(\frac{\Omega_0\tau_2}{2}) - \cos^4(\frac{\Omega_0\tau_2}{2})]\sin(\delta\phi_0)}{\sin^4(\frac{\Omega_0\tau_2}{2})\cos(\delta\phi_0) - \frac{1}{2}\sin^2(\Omega_0\tau_2)\cos(2\delta\phi_0) + \cos^4(\frac{\Omega_0\tau_2}{2})\cos(\delta\phi_0)}. \quad (37)$$

In the limit where $\delta\phi_0$ is small, Eq. (37) reduces to:

$$\Delta\phi = -\tan^{-1}\frac{\delta\phi_0}{\sin(\Omega_0\delta\tau)}. \quad (38)$$

As expected, we see that in the limits of $\delta\tau = \pm\,\tau/2$, the fringe shift approaches the value of $\mp\,\delta\phi_0$ in Eq. (33). Therefore, in these limits, $A_{\text{eff}} = A_0$. As $|\delta\tau|$ becomes smaller than $\tau/2$, the magnitude of the fringe shift actually becomes bigger than $\delta\phi_{\,0}$. However, this does not represent any improvement in our ability to measure the rate of rotation, due to the fact that the fringe amplitudes become smaller at the same time. This can be appreciated immediately by noting that in the limit of $\delta\tau \to 0$, the fringe shift approaches $\pi/2$, independent of the rate of rotation, but the fringe amplitude approaches zero.

In order to interpret this result quantitatively, it is instructive to define a *minimum measurable rotation rate*: Ω_{mm}. By rearranging Eq. (33), we see that Ω_{mm} depends on the minimum measurable fringe shift $\Delta\phi_{\text{mm}}$: