Sum-Capacity of a Class of K-user Gaussian Interference Channels within $O(K)$ bits

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***Abstract*—Though the capacity of the 2-user Gaussian interference channel has long eluded information theorists, recent progress has been made by focussing on approximations with provable bounds. However, extensions to a general K-user network has proven to be non-obvious, in particular due to the role of interference alignment in these cases. In this paper, we look at a special case of a K-user Gaussian interference network where only one of the users interferes with and is also interfered by all the other users. We determine the sum-capacity of such a network within $O(K)$ bits for all possible values of the channel parameters, provided the direct signal is stronger than the receiver noise.**

I. Introduction

Starting with the work of Etkin-Tse-Wang [8], there has been an increased interest in studying constant gap approximations to capacity regions for many long-standing problems in network information theory. In [8] this approach is applied to the 2-user interference channel (IC) where the capacity is approximated by deriving new tighter upperbounds and achievability shown by specializing a particular Han-Kobayashi [1] power splitting scheme using Gaussian code books. This result is closely related to the analysis of the 2-user linear deterministic interference channel [2] that in some way approximates the Gaussian counterpart. The linear deterministic model, first introduced in [3] has been a useful tool to gain understanding into how signals interact in a multi-terminal network, ignoring the effect of receiver noise. The cases where the exact capacity of this approximate channel can be computed with relative ease often lead to constant-gap approximations of the capacity in the corresponding Gaussian channels [4]–[7], [9], [13].

The results in [8] do not generalize naturally to interference channels with $K > 2$ users. This is essentially due to the fact that signals from more than one transmitter superimpose at a receiver. When this interference has structure, it can be exploited. This is demonstrated in [11] for a particular 3-user Gaussian IC, where a higher rate is shown to be achievable using a layered lattice coding scheme as opposed to a Han-Kobayashi scheme as in [8]. The structure provided by lattice codes enables a form of interference alignment [10]. However, in general, IC capacity characterization in many-user scenarios has been met with limited success. Some partial results exist either in specific regimes of channel gains [16] or for some restricted classes of channels [17]. In [12], the approximate capacity for a K user Gaussian IC is provided for two special cases - ‘one-to-many’ and ‘many-to-one’, where all interference is either caused by or is caused to only one user, respectively. The results again make use of the exact analysis of the corresponding deterministic case. Even for some linear deterministic 3-user ICs with specific connectivity [14], [15], sum-capacity itself is not fully characterized. Recently, the approximate capacity was derived for a K-user Gaussian IC with specific cyclic interference pattern [19]. However, in this kind of interference channel, the key problem of interference alignment does not arise. A new achievable rate region evaluated for a fully-connected 3-user deterministic IC by ‘interference decoding’ [18] offers new insights into this problem.

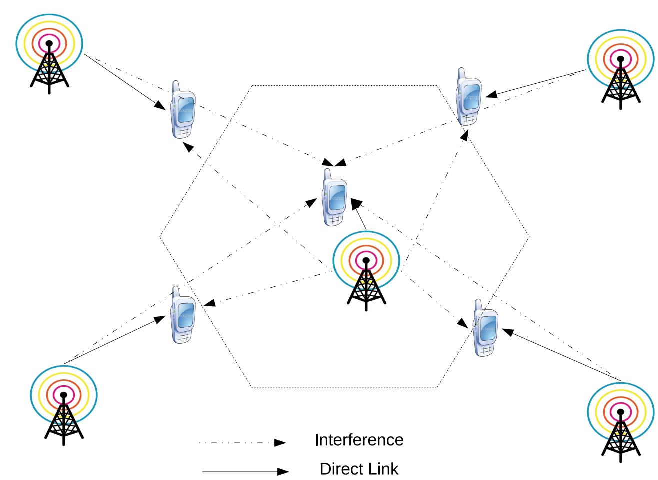

Fig. 1. Mobiles near the boundary of a cell

In this paper, we consider a very simple class of K-user Gaussian IC in which only one of the users causes interference to all others, as well as is interfered by all of them. This network can be looked upon as a superposition of a one-to-many and many-to-one networks as in [12] and incorporates the tension between multiple signals interfering at a receiver as well as the same signal interfering at multiple receivers. Apart from its usefulnes in throwing light on the general K-user interference network, this network models a situation for mobile communication where a number of mobile terminals are near the boundary of a particular cell as illustrated in Figure 1. Based on this configuration of the interfering users, we call this a 'star' IC. For tractability, we only consider a fully-symmetric setup (all cross channels have the same parameter value) and seek to find an approximate sum capacity for this network. This builds on our work in [15], where we characterized the sum-capacity of the corresponding 3-user linear deterministic IC. Once again, this analysis provides insights for approximating the Gaussian case.

The key contributions of this paper are:

- We characterize the sum-capacity of a symmetric K-user Gaussian star IC within $O(K)$ bits.
- We show that a single class of strategies might not be optimal in all interference regimes and judicious choice of codebooks and power control is required in our achievable scheme.
- We demonstrate that it is possible for a non-trivially connected K-user IC to achieve more than $K/2$ spatial degrees of freedom, which is interesting when compared with the findings in [23] where it was shown that degrees of freedom for a K-user Gaussian IC with non-zero rational coefficients is strictly smaller than $K/2$.

The rest of the paper is organized as follows: Section II introduces the system model, some useful upperbounds to the sum-capacity are derived in Section III, coding schemes that approximately achieve sum-capacity are discussed in Section IV and concluding remarks and future directions are provided in Section V.

II. System Model

A K-user symmetric Gaussian star IC as shown in Figure 2 has K transmitter-receiver pairs. Each transmitter tries to communicate with its intended receiver. In this process, only transmitter 1 causes interference to all other receivers, while all the other transmitters in turn, cause interference to receiver 1. Let X_i and Y_i denote the input and output signal of the ith user respectively, while $Z_i \sim \mathcal{CN}(0,1)$ is independent and identically distributed Gaussian noise that impairs receiver i. Each X_i has an associated power constraint P so that $\mathbb{E}[|X_i|^2] \leq P$. Let h denote the cross-channel gain, whereas, the direct-channel gain is normalized to 1. These gains are assumed to be same for all users. The interference network can be formally specified by the following input-output relationships:

$$\begin{aligned} Y_1 &= X_1 + hX_2 + hX_3 + \ldots hX_K + Z_1 \\ Y_i &= hX_1 + X_i + Z_i, \qquad \forall i = 2,3,\ldots K. \end{aligned} \tag{1}$$

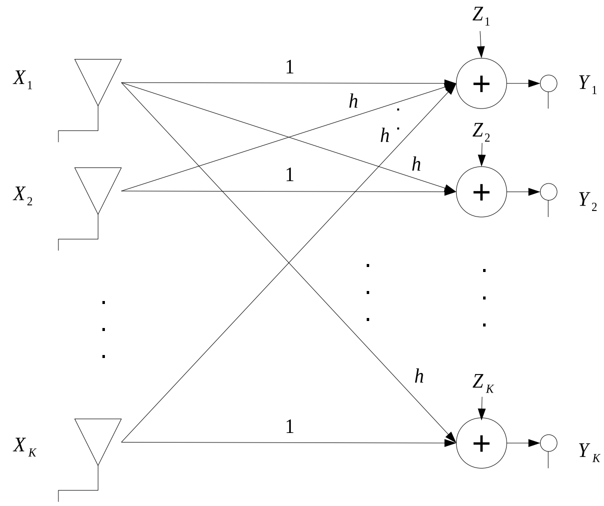

Fig. 2. A K-user symmetric Gaussian 'star' IC

We define two quantities, signal-to-noise ratio and interference to noise ratio as follows:

$$\mathsf{SNR} = P, \qquad \mathsf{INR} = |h|^2 P. \tag{2}$$

Depending on the strength of the cross-links, the K-user interference network can be deemed to be in the weak interference regime if $|h|^2 \leq 1$ or in the strong interference regime if $|h|^2 > 1$. Note that, in this paper we are concerned only with situations where the direct signal is stronger than the receiver noise ($\mathsf{SNR} > 1$). For cases when $\mathsf{INR} > 1$, we introduce a parameter $\alpha > 0$ defined by $\mathsf{INR} = P^{\alpha}$; this α parameter is often used to specify the corresponding linear deterministic model. Reducing to the case $K = 2$ gives us the symmetric 2-user interference channel, while $K = 3$ gives the symmetric shoe-string interference channel from [15].

III. Upper Bounds

In this section we derive several upperbounds on the sum-rate achievable in the K-user Gaussian star IC discussed before. These will be used later to show constant bit-gap results for sum-capacity.

Theorem 1: For the K-user Gaussian star IC in (1), the sum-capacity in the weak interference regime is upperbounded by the minimum of the following three quantities:

$$R_{ub1}^{weak} = \log\left(1 + \frac{\mathsf{SNR}}{1+\mathsf{INR}}\right) + (K-1)\log\left(1+\mathsf{SNR}\right) \tag{3}$$

$$\begin{aligned} R_{ub2}^{weak} = \log&\left(1 + (K-1)\,\mathsf{INR} + \frac{\mathsf{SNR}}{1+\mathsf{INR}}\right) + \\ &\log\left(1 + \mathsf{INR} + \frac{\mathsf{SNR}}{1+\mathsf{INR}}\right) + \\ &(K-2)\log\left(1+\mathsf{SNR}\right) \end{aligned} \tag{4}$$

$$R_{ub3}^{weak} = \log\left(1+\mathsf{INR}+\mathsf{SNR}\right) + \log\left(1+2\mathsf{INR}+\frac{\mathsf{SNR}}{1+\mathsf{INR}}\right) + \log\left(\frac{1+\mathsf{SNR}}{1+\mathsf{INR}}\right) + (K-3)\log\left(1+\mathsf{SNR}\right). \quad (5)$$

Proof: First, note the similarity of the form of the upperbounds with that of the upperbounds for the symmetric 2-user Gaussian IC in [8]. In fact, the first and the third upperbounds are obtained by simply extending the upperbounds as derived in [8].

To show that R_{ub1}^{weak} is an upperbound to the sum-capacity for our K-user network, consider removing all cross-links except those between users 1 and 2. Clearly, any sum-rate upperbound on this new network should also be an upperbound on the original one as presence of interference can only reduce the desired rate. Now, the new network can essentially be decomposed into a symmetric 2-user Gaussian IC and $(K-2)$ parallel point-to-point AWGN channels. Using the known upperbounds for each of these quantities, we easily arrive at the first upperbound.

For the second upperbound, we consider a side-information converse akin to that given in Theorem 1, [8]. Define

$$\begin{aligned} S_1 &= hX_1 + Z_2 \\ S_2 &= hX_2 + Z_1 \\ S_3 = S_4 &= \dots S_K = X_1. \end{aligned} \quad (6)$$

and consider a genie-aided channel where a genie provides S_i to receiver i. Clearly, the capacity region of this genie-aided channel is an upperbound to the capacity region of the original interference channel. Therefore, we can obtain an upperbound for the sum-rate of the original channel by computing an upperbound on the sum-rate of the genie-aided channel. For a block of length n, Fano's inequality gives

$$\begin{aligned} & n(R_1 + R_2 + \dots + R_K) \\ & \le \sum_{i=1}^{K} I(X_i^n; Y_i^n, S_i^n) + n\epsilon_n \\ & = \sum_{i=1}^{K} I(X_i^n; S_i^n) + I(Y_i^n; X_i^n|S_i^n) + n\epsilon_n \\ & = \sum_{i=1}^{2} h(S_i^n) - h(S_i^n|X_i^n) + h(Y_i^n|S_i^n) - h(Y_i^n|S_i^n, X_i^n) \\ & + \sum_{i=3}^{K} h(Y_i^n|S_i^n) - h(Y_i^n|S_i^n, X_i^n) + n\epsilon_n \\ & = h(S_1^n) - h(Z_2^n) + h(Y_1^n|S_1^n) - h(S_2^n + \sum_{i=3}^{K} hX_i) \\ & + h(S_2^n) - h(Z_1^n) + h(Y_2^n|S_2^n) - h(S_1^n) \\ & + \sum_{i=3}^{K} h(X_i^n + Z_i^n) - h(Z_i^n) + n\epsilon_n \\ & \le h(Y_1^n|S_1^n) - h(Z_1^n) + h(Y_2^n|S_2^n) - h(Z_2^n) \end{aligned} \quad (7)$$

$$\begin{aligned} & + \sum_{i=3}^{K} h(X_i^n + Z_i^n) - h(Z_i^n) + n\epsilon_n \\ & \le \sum_{t=1}^{n} \big[h(Y_{1t}|S_{1t}) - h(Z_{1t}) + h(Y_{2t}|S_{2t}) - h(Z_{2t}) \quad (8) \\ & + \sum_{i=3}^{K} h(X_{it} + Z_{it}) - h(Z_{it}) \big] + n\epsilon_n \quad (9) \end{aligned}$$

where, (8) from the fact that $h(S_2^n) - h(S_2^n + \sum_{i=3}^{K} hX_i) \le 0$ while (9) follows by the fact that removing conditioning cannot reduce differential entropy and that Gaussian noise is independent. Also note that $\epsilon_n \to 0$ as $n \to \infty$. Let $\mathbb{E}[|X_{it}|^2] = P_{it}$ such that $\sum_{t=1}^{n} P_{it} \le nP$. Now, for $i = 1, 2$, $h(Y_{it}|S_{it}) \le \log(2\pi e \text{Var}(Y_{it} - \alpha S_{it}))$ where $\alpha = \frac{\mathbb{E}[Y_{it} S_{it}^*]}{\mathbb{E}[S_{it} S_{it}^*]}$ as shown in [20]. Thus we obtain,

$$\begin{aligned} \frac{1}{n}\sum_{t=1}^{n} h(Y_{1t}|S_{1t}) & \le \frac{1}{n}\log\left[\pi e\left(1 + |h|^2 \sum_{i=2}^{K} P_{it} + \frac{P_{1t}}{1+|h|^2 P_{1t}}\right)\right] \\ & \le \log\left[\pi e\left(1 + |h|^2 (K-1)\left(\frac{1}{n}\sum_{t=1}^{n} P_{1t}\right) + \frac{\left(\frac{1}{n}\sum_{t=1}^{n} P_{1t}\right)}{1+|h|^2\left(\frac{1}{n}\sum_{t=1}^{n} P_{1t}\right)}\right)\right] \\ & \le \log\left[\pi e\left(1 + |h|^2 (K-1)P + \frac{P}{1+|h|^2 P}\right)\right] \end{aligned} \quad (10)$$

where the second step follows by Jensen's inequality applied to a concave function and the last step is due to the fact the function is increasing in P. Similar calculations yield

$$\frac{1}{n}\sum_{t=1}^{n} h(Y_{2t}|S_{2t}) \le \log\left[\pi e\left(1 + |h|^2 P + \frac{P}{1+|h|^2 P}\right)\right]. \quad (11)$$

Further, noting that the Gaussian distribution maximizes differential entropy under a given variance constraint so that $h(X_{it} + Z_{it}) \le \log\left(\pi e(P_{it} + 1)\right)$ and applying Jensen's inequality again, we get

$$\frac{1}{n}\sum_{t=1}^{n}\sum_{i=3}^{K} h(X_{it} + Z_{it}) \le (K-2)\log\left(\pi e(1+P)\right). \quad (12)$$

Finally, using equations (9), (10), (11), (12) and the fact that the noise has variance 1, we get the desired upperbound R_{ub2}^{weak}.

For the third upperbound, we first consider a network where all the crosslinks between user 1 and users $4, 5, \dots K$ have been removed. Clearly, any upperbound on the sum-capacity of this channel (a 3-user IC - where user 1 is interfered by and also interferes with users 2 and 3, and $K-3$ parallel point-to-point channels) is also an upperbound on the sum-capacity of the original channel. First we look at this 3-user IC. Let the genie provide side-information $S_1 = hX_1 + Z_3$ to receiver

1 and $S_2 = X_1$ to receiver 2. Then, considering a block of length n, Fano's inequality gives

$$
\begin{aligned}
& n(R_1+R_2+R_3) \\
& \le I(X_1^n;Y_1^n,S_1^n)+I(X_2^n;Y_2^n|X_1^n)+I(X_3^n;Y_3^n)+n\epsilon_n \\
& = I(X_1^n;S_1^n)+I(X_1^n;Y_1^n|S_1^n)+I(X_2^n;Y_2^n|X_1^n) \\
& + I(X_3^n;Y_3^n)+n\epsilon_n \\
& = h(S_1^n)-h(S_1^n|X_1^n)+h(Y_1^n|S_1^n)-h(Y_1^n|S_1^n,X_1^n)+ \\
& h(Y_2^n|X_1^n)-h(Y_2^n|X_1^n,X_2^n)+h(Y_3^n)-h(Y_3^n|X_3^n)+n\epsilon_n \\
& = h(Y_3^n)+h(Y_1^n|S_1^n)-h(Z_2^n)-h(Z_3^n) \\
& - h(hX_2^n+hX_3^n+Z_1^n)+h(X_2^n+Z_2^n)+n\epsilon_n \\
& \le h(Y_3^n)+h(Y_1^n|S_1^n)-h(Z_2^n)-h(Z_3^n) \\
& - h(hX_2^n+Z_1^n)+h(X_2^n+Z_2^n)+n\epsilon_n \\
& \le \sum_{t=1}^{n}[h(Y_{3t})+h(Y_{1t}|S_{1t})-h(Z_{2t})-h(Z_{3t})] \qquad (13) \\
& - h(hX_2^n+Z_1^n)+h(X_2^n+Z_2^n)+n\epsilon_n \qquad (14)
\end{aligned}
$$

where (13) follows from the fact that $h(hX_2^n+Z_1^n) = h(hX_2^n+hX_3^n+Z_1^n|X_3^n) \le h(hX_2^n+hX_3^n+Z_1^n)$. Now, since in the low interference regime $|h|<1$, by the worst case noise result [21], we get the following upperbound:

$$
-h(hX_2^n+Z_1^n)+h(X_2^n+Z_2^n) \le \log\left(\frac{1+\mathsf{SNR}}{1+\mathsf{INR}}\right). \qquad (15)
$$

Also, as in the proof of the second upperbound, we have,

$$
\frac{1}{n}\sum_{t=1}^{n} h(Y_{1t}|S_{1t}) \le \log\left[\pi e\left(1+2|h|^2P+\frac{P}{1+|h|^2P}\right)\right]. \qquad (16)
$$

Further, $h(Y_{3t})$ is maximized when $X_{1t}, X_{3t} \sim \mathcal{CN}(0,P)$ so that

$$
\frac{1}{n}\sum_{t=1}^{n} h(Y_{3t}) \le \log\left[\pi e\left(1+|h|^2P+P\right)\right]. \qquad (17)
$$

Using (14),(15),(16),(17) and the point-to-point capacity bounds for each of the users $4,\dots K$, we get R_{ub3}^{weak} as the desired upperbound on the sum-rate. ∎

Theorem 2: For the K-user Gaussian symmetric star IC, the sum-capacity in the strong interference regime is upper-bounded by the minimum of the following two quantities:

$$
R_{ub1}^{strong} = \log\left(1+\mathsf{INR}+\mathsf{SNR}\right)+(K-2)\log\left(1+\mathsf{SNR}\right) \qquad (18)
$$

$$
R_{ub2}^{strong} = K\log\left(1+\mathsf{SNR}\right). \qquad (19)
$$

Proof: In the strong interference regime, to prove the first upperbound, consider a channel where all crosslinks between user 1 and users $3,4,\dots K$ have been removed. Any upperbound to this reduced channel (a 2-user IC and $K-2$ parallel point-to-point channels) is also an upperbound to the original channel. In the strong interference regime, a 2-user IC must satisfy MAC constraint at each receiver, so that $\log(1+\mathsf{INR}+\mathsf{SNR})$ is an upperbound to R_1+R_2. Combining this with the point-to-point constraint of the other $K-2$ parallel channels, we get R_{ub1}^{strong}.

On the otherhand, R_{ub2}^{strong} is simply obtained by combining the individual point-to-point rate constraints of all the users. ∎

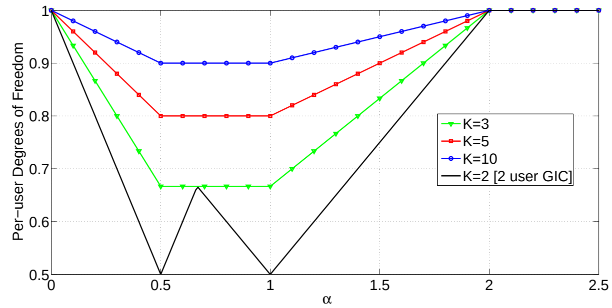

Fig. 3. Per-user generalized degrees of freedom for the symmetric star GIC

As in [8], we illustrate these bounds by plotting the corresponding per-user generalized degrees of freedom as a function of the paramter α for different values of K. We also include the corresponding curve from [8] for $K=2$, which is a symmetric fully connected GIC. In the next section we show that these degrees of freedom are achievable. Examining Fig. 3, several observations can be made. First, for $K>2$ the corresponding curve has a "U" shape instead of the "W" shape seen for $K=2$. Second for $\alpha<2$ the degrees of freedom per user are increasing with K. Essentially, compared to $K=2$, the sum-rate optimal strategy is giving more rate to users $i\ge 2$ and reducing the rate of user 1. As K increases, there are more of these users and their share of the total rate is increasing.

IV. Achievability

The main result of this paper is stated in the theorem below:

Theorem 3: For the K-user symmetric Gaussian star IC, a sum-rate within $O(K)$ bits of the bounds in Theorems 1 and 2 is achievable.

Proof: The achievability (upto $O(K)$ bits) will be proved separately in different parameter regimes.

A. Weak interference

1) $\mathsf{INR}<1$*:* In this regime, in the deterministic case, all users transmit independent bits from interference free levels. In the Gaussian case, an achievable scheme is to allow all K transmitters to use Gaussian codebooks and transmit signals using full power P. At the receivers, they treat interference as noise. Clearly, the achievable sum-rate is $R_{achiev1}^{weak} = (K-1)\log\left(1+\frac{\mathsf{SNR}}{1+\mathsf{INR}}\right)+\log\left(1+\frac{\mathsf{SNR}}{1+(K-1)\mathsf{INR}}\right)$. Comparing this with R_{ub1}^{weak}, we get,

$$\begin{aligned}
&\delta_1^{weak} \\
&= R_{ub1}^{weak} - R_{achiev1}^{weak} \\
&= (K-1)\log\left(\frac{1+\mathsf{SNR}}{1+\frac{\mathsf{SNR}}{1+\mathsf{INR}}}\right) + \log\left(\frac{1+\frac{\mathsf{SNR}}{1+\mathsf{INR}}}{1+\frac{\mathsf{SNR}}{1+(K-1)\mathsf{INR}}}\right) \\
&< (K-1)\log\left(\frac{2(1+\mathsf{SNR})}{(2+\mathsf{SNR})}\right) + \log\left(\frac{K(1+\mathsf{SNR})}{(K+\mathsf{SNR})}\right) \\
&< (K-1) + \log K \\
&= O(K). \qquad (20)
\end{aligned}$$

2) $1 \le \mathsf{INR} < \sqrt{\mathsf{SNR}}$: ($0 \le \alpha < \frac{1}{2}$): For the deterministic case, stopping user 1 from transmitting from any level, while allowing other users to use all the available levels is optimum. But such a scheme would not be optimal in the Gaussian case. Here, we consider a type of a power controlled transmission scheme. All transmitters except transmitter 1 use Gaussian codebooks and transmit signals using full power P. On the other hand, transmitter 1 transmits at a power $P_1 = P^{1-\alpha}$. At the receivers, all interference is treated as noise. Clearly, the achievable sum-rate is

$$\begin{aligned}
R_{achiev2}^{weak} &= (K-1)\log\left(1+\frac{P}{1+|h|^2P_1}\right) \\
&\quad + \log\left(1+\frac{P_1}{1+(K-1)|h|^2P}\right) \\
&= (K-1)\log\left(1+\frac{\mathsf{SNR}}{2}\right) \\
&\quad + \log\left(1+\frac{\mathsf{SNR}}{\mathsf{INR}(1+(K-1)\mathsf{INR})}\right). \qquad (21)
\end{aligned}$$

Comparing this with R_{ub2}^{weak} we get,

$$\begin{aligned}
\delta_2^{weak} &= R_{ub2}^{weak} - R_{achiev2}^{weak} \\
&< (K-1) + \log(K(K-1)) + \log 3 \\
&= O(K). \qquad (22)
\end{aligned}$$

Note that this scheme achieves all the $K-2\alpha$ available degrees of freedom.

3) $\sqrt{\mathsf{SNR}} \le \mathsf{INR} < \mathsf{SNR}$: ($\frac{1}{2} \le \alpha < 1$): Here also power control is used essentially to silence transmitter 1, which is what is directly obtained from the deterministic case. All transmitters except transmitter 1 use Gaussian codebooks and transmit signals using full power P. On the other hand, transmitter 1 does not transmit at all. At the receivers, all interference is treated as noise. Clearly, the achievable sum-rate is $R_{achiev3}^{weak} = (K-1)\log(1+\mathsf{SNR})$. Comparing this with R_{ub3}^{weak} we get a constant bit gap as shown below:

$$\begin{aligned}
\delta_3^{weak} &= R_{ub3}^{weak} - R_{achiev3}^{weak} \\
&= \log(1+P+P^\alpha) + \log\left(1+2P^\alpha+\frac{P}{1+P^\alpha}\right) \\
&\quad - \log(1+P^\alpha) \\
&= \log(1+\frac{P^\alpha}{1+P}) + \log\left(2+\frac{P}{(1+P^\alpha)^2}\right) \\
&< \log(1+P^{\alpha-1}) + \log(2+P^{1-2\alpha}) \\
&< \log 2 + \log 3 \\
&= O(1) \qquad (23)
\end{aligned}$$

where the second to last step follows from the fact that $\alpha-1 < 0$ and $1-2\alpha < 0$ in this regime. All the available $K-1$ degrees of freedom can be achieved using this strategy.

B. Strong interference

In this regime, the idea of interference alignment becomes important and lattice coding techniques are used to show (approximate) achievability.

1) $\mathsf{SNR} \le \mathsf{INR} < \mathsf{SNR}\sqrt{\mathsf{SNR}}$: ($1 \le \alpha < \frac{3}{2}$): We consider a nested lattice scheme inspired by the one used in [12] to replicate the strategy for the deterministic IC in [15]. In the deterministic case, all the users except user 1 transmit independent bits from all the available levels, while user 1 transmits at a lower rate and uses some of its available levels judiciously to repeat a part of its information bits that allows other users to cancel the interference. Also, it does not use some of the middle levels to transmit any bit. We use a similar idea in the corresponding Gaussian case. The signal power as observed at receiver 1 is partitioned into the following levels: $P^\alpha, P, P^{2-\alpha}, P^{\alpha-1}, 1, 0$. Note that, for $\alpha > 1$, the levels are in a decreasing order of magnitude. Define

$$\begin{aligned}
\theta_1 &= P^\alpha - P \\
\theta_2 &= P - P^{2-\alpha} \\
\theta_3 &= P^{2-\alpha} - P^{\alpha-1}. \qquad (24)
\end{aligned}$$

All the users except user 1 decompose the transmitted signal into a sum of three independent components given by $X_i = \sum_{k=1}^{3} X_i(k)$; component $X_i(k)$ being the user i's input to the k^{th} transmit level. The signal $X_i(k)$ is transmitted with a power $\frac{\theta_k}{|h|^2} = \theta_k P^{1-\alpha}$, so that, at receiver 1 where each of these signals interfere, it has a power θ_k.

At the receivers of all the other users $2, 3, \ldots K$, the observed signal power is split into the following levels: $P^\alpha, P, P^{2(\alpha-1)}, P^{\alpha-1}, 1, 0$. Here, $1 \le \alpha < \frac{3}{2}$ ensures that the levels are in order of magnitude. Again define

$$\begin{aligned}
\phi_1 &= P^\alpha - P \\
\phi_2 &= P - P^{2(\alpha-1)} \\
\phi_3 &= P^{2(\alpha-1)} - P^{\alpha-1}. \qquad (25)
\end{aligned}$$

As before, user 1 also decomposes its transmitted signal into components with a power $\frac{\phi_k}{|h|^2} = \phi_k P^{1-\alpha}$, so that, at all the receivers where this signal interferes, it has a power ϕ_k.

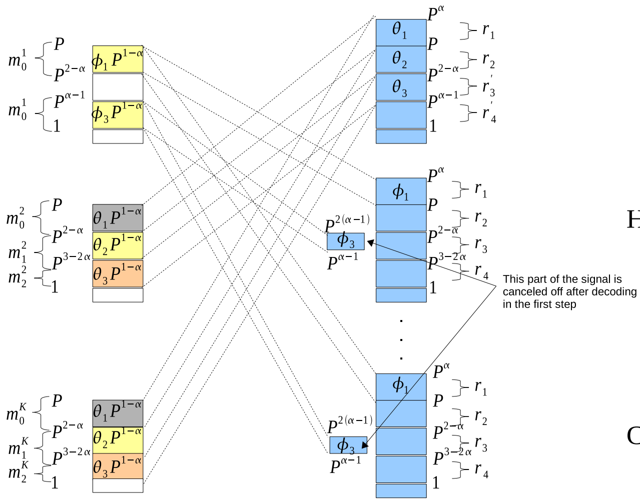

Fig. 4. Power-split for lattice codes when $\alpha < \frac{3}{2}$

However, user 1 assigns no power to the second level, so that $X_1 = X_1(1) + X_1(3)$. Further, user 1 transmits the same message from both the levels it seeks to use. Note that while all users $2, 3, \dots K$ satisfy the power constraint with equality, user 1 transmits at a lower power.

For each θ_k and ϕ_k that is used, a lattice code is selected as in [22] such that the spherical shaping region has an average power per dimension θ_k and ϕ_k, respectively, and the lattice is good for channel coding. Further, the same code is chosen for θ_1 and ϕ_1 to ensure alignment. The rate of the lattice is chosen to allow decoding of encoded messages as illustrated in Figure 4.

Next we describe the decoding procedure at each receiver. Decoding occurs from the top level downwards, treating the signals from lower level as Gaussain noise. When the signal on a level is decoded, it is subtracted off completely, and decoding proceeds with the next highest level. At any level, an aggregate signal is decoded first by decoding to the nearest lattice point. By making use of the available results in lattice coding for Gaussian channels [22] and the fact that user 1 transmits the same message at both levels 1 and 3, we arrive at the following rate bounds for decodability:

$$\begin{aligned}
r_1 &\le \log\left(\frac{P^\alpha - P}{1+P}\right)\\
r_2 &\le \log\left(\frac{P - P^{2-\alpha}}{1+P^{2-\alpha}}\right)\\
r_3 &\le \log\left(\frac{P^{2-\alpha} - P^{3-2\alpha}}{1+P^{3-2\alpha}}\right)\\
r_4 &\le \log\left(P^{3-2\alpha} - 1\right)\\
r_3^{'} &\le \log\left(\frac{P^{2-\alpha} - P^{\alpha-1}}{1+P^{\alpha-1}}\right)\\
r_4^{'} &\le \log\left(P^{\alpha-1} - 1\right)
\end{aligned} \tag{26}$$

which relate to the user rates as follows:

$$\begin{aligned}
&R_2 = R_3 = \dots = R_K = \min(r_1, r_2) + \min(r_2, r_3)\\
&+ \min(r_3^{'}, r_4)\\
&= r_2 + r_3 + r_4\\
&R_1 = \min(r_1, r_2, r_4^{'}) = r_4^{'}.
\end{aligned} \tag{27}$$

Hence, the achievable sum-rate is given by,

$$\begin{aligned}
&R_{achiev1}^{strong} =\\
&(K-1)\log\left(\left(\frac{P - P^{2-\alpha}}{1+P^{2-\alpha}}\right)\left(\frac{P^{2-\alpha} - P^{3-2\alpha}}{1+P^{3-2\alpha}}\right)\right.\\
&\left.\left(P^{3-2\alpha} - 1\right)\right) + \log\left(P^{\alpha-1} - 1\right).
\end{aligned} \tag{28}$$

Comparing this with R_{ub1}^{strong}, we get,

$$\begin{aligned}
&\delta_1^{strong}\\
&= R_{ub1}^{strong} - R_{achiev1}^{strong}\\
&= \log(1 + P^\alpha + P) + (K-2)\log(1+P)\\
&- (K-1)\log\left(\left(\frac{P - P^{2-\alpha}}{1+P^{2-\alpha}}\right)\left(\frac{P^{2-\alpha} - P^{3-2\alpha}}{1+P^{3-2\alpha}}\right)\right.\\
&\left.(P^{3-2\alpha} - 1)\right) - \log(P^{\alpha-1} - 1)\\
&< \log(3P^\alpha) + (K-2)\log(2P) + (K-1)\log(8)\\
&- (K-1)\log\left(\left(\frac{1+P}{1+P^{2-\alpha}}\right)\left(\frac{1+P^{2-\alpha}}{1+P^{3-2\alpha}}\right)(P^{3-2\alpha})\right)\\
&- \log(P^{\alpha-1} - 1)\\
&< \log 3 + \log(P^\alpha) + (K-2) + (K-2)\log(P) + 3K\\
&- 3 - (K-1)\log\left(\left(\frac{P}{2P^{2-\alpha}}\right)\left(\frac{P^{2-\alpha}}{2P^{3-2\alpha}}\right)(P^{3-2\alpha})\right)\\
&- \log(P^{\alpha-1}) + 1\\
&= \log 3 + 6(K-1) + \log(P^\alpha) + (K-2)\log(P)\\
&- (K-1)\log(P) - \log(P^{\alpha-1})\\
&= \log 3 + 6(K-1)\\
&= O(K).
\end{aligned} \tag{29}$$

2) $\mathsf{SNR}\sqrt{\mathsf{SNR}} \le \alpha < \mathsf{SNR}^2$*:* ($\frac{3}{2} \le \alpha < 2$): A similar nested lattice scheme is employed in this regime as well. In this regime too, the achievable strategy of the deterministic case provides us the clue to achievability in the Gaussian case. The achievable strategy is similar to the previous regime, except that now the unused levels of user 1 are used to transmit independent information bits. In the corresponding Gaussian case, at *all* the receivers, power is partitioned into the following levels: $P^\alpha, P^{2(\alpha-1)}, P, P^{\alpha-1}, P^{2-\alpha}, 1, 0$. Note that, for $2 > \alpha \ge \frac{3}{2}$, this is a decreasing order. As before, define

$$\begin{aligned}
\theta_1 &= P^\alpha - P^{2(\alpha-1)}\\
\theta_2 &= P^{2(\alpha-1)} - P\\
\theta_3 &= P - P^{\alpha-1}\\
\theta_4 &= P^{\alpha-1} - P^{2-\alpha}
\end{aligned} \tag{30}$$

$$\theta_5 = P^{2-\alpha} - 1. \tag{31}$$

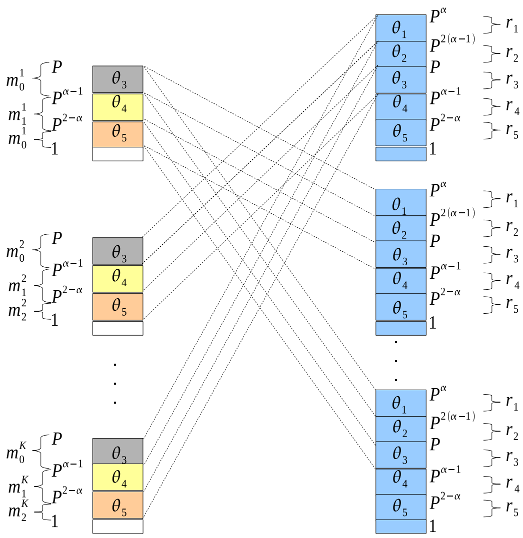

Fig. 5. Power-split for lattice codes when $2 > \alpha > \frac{3}{2}$

As illustrated in Fig. 5, signals transmitted with power θ_i interfere other users at power level θ_{i+2}. The lattice codes used for transmission at each power level (from θ_3 to θ_5) are different from each other, but same for all users. Further, all users except user 1 transmits independent messages from the different levels but user 1 transmits the same message from θ_3 and θ_5.

By similar arguments as before, we arrive at the following rate bounds for decodability:

$$
\begin{aligned}
r_1 &\leq \log\left(\frac{P^\alpha - P^{2(\alpha-1)}}{1+P^{2(\alpha-1)}}\right)\\
r_2 &\leq \log\left(\frac{P^{2(\alpha-1)} - P}{1+P}\right)\\
r_3 &\leq \log\left(\frac{P - P^{\alpha-1}}{1+P^{\alpha-1}}\right)\\
r_3 &\leq \log\left(\frac{P^{\alpha-1} - P^{2-\alpha}}{1+P^{2-\alpha}}\right)\\
r_5 &\leq \log\left(P^{2-\alpha}-1\right)
\end{aligned} \tag{32}
$$

which relate to the user rates as follows:

$$
\begin{aligned}
&R_2 = R_3 = \cdots = R_K = \min(r_1, r_3) + \min(r_2, r_4) + r_5\\
&= r_3 + r_4 + r_5\\
&R_1 = \min(r_1, r_3, r_5) + \min(r_2, r_4) = r_4 + r_5.
\end{aligned} \tag{33}
$$

Hence the achievable sum-rate is given by,

$$
\begin{aligned}
&R_{achiev2}^{strong} =\\
&(K-1)\log\left(\left(\frac{P-P^{\alpha-1}}{1+P^{\alpha-1}}\right)\left(\frac{P^{\alpha-1}-P^{2-\alpha}}{1+P^{2-\alpha}}\right)\right.\\
&\left.\left(P^{2-\alpha}-1\right)\right) + \log\left(\left(\frac{P^{\alpha-1}-P^{2-\alpha}}{1+P^{2-\alpha}}\right)\left(P^{2-\alpha}-1\right)\right).
\end{aligned}
$$

As before, comparing this with R_{ub1}^{strong}, we get,

$$
\begin{aligned}
&\delta_2^{strong}\\
&= R_{ub1}^{strong} - R_{achiev2}^{strong}\\
&= \log(1+P^\alpha+P) + (K-2)\log(1+P)\\
&(K-1)\log\left(\left(\frac{P-P^{\alpha-1}}{1+P^{\alpha-1}}\right)\left(\frac{P^{\alpha-1}-P^{2-\alpha}}{1+P^{2-\alpha}}\right)\right.\\
&\left.\left(P^{2-\alpha}-1\right)\right) - \log\left(\left(\frac{P^{\alpha-1}-P^{2-\alpha}}{1+P^{2-\alpha}}\right)\left(P^{2-\alpha}-1\right)\right)\\
&< \log(3P^\alpha) + (K-2)\log(2P) + (K-1)\log(8)\\
&-(K-1)\log\left(\left(\frac{1+P}{1+P^{\alpha-1}}\right)\left(\frac{1+P^{\alpha-1}}{1+P^{2-\alpha}}\right)\left(P^{2-\alpha}\right)\right)\\
&-\log\left(\left(\frac{1+P^{\alpha-1}}{1+P^{2-\alpha}}\right)\left(P^{2-\alpha}\right)\right) + \log 4\\
&< \log 3 + \log(P^\alpha) + (K-2) + (K-2)\log(P) + 3K\\
&-3-(K-1)\log\left(\left(\frac{P}{2P^{\alpha-1}}\right)\left(\frac{P^{\alpha-1}}{2P^{2-\alpha}}\right)\left(P^{2-\alpha}\right)\right)\\
&-\log\left(\left(\frac{P^{\alpha-1}}{2P^{2-\alpha}}\right)\left(P^{2-\alpha}\right)\right) + \log 4\\
&= 2 + \log 3 + 6(K-1) + \log(P^\alpha) + (K-2)\log(P)\\
&-(K-1)\log(P) - \log(P^{\alpha-1})\\
&= 2 + \log 3 + 6(K-1)\\
&= O(K).
\end{aligned} \tag{34}
$$

Note that for the entire range $1 < \alpha < 2$, all the $K+\alpha-2$ degrees are freedom are achievable by doing interference alignment using lattice codes.

3) $\mathsf{INR} \geq \mathsf{SNR}^2$: ($\alpha \geq 2$): The deterministic case strategy is simply to ignore the interference as all of it sits above the signal level of each user. Correspondingly, in the Gaussian case in this regime, we use a very simple strategy where all the users transmit at full power using the same lattice code. At each receiver, first the total interference is decoded, treating its own signal as noise; then the decoded interference is canceled out to decode its own signal. Clearly then, $R_i = \min\left(\log\left(\frac{\mathsf{INR}}{1+\mathsf{SNR}}\right), \log(\mathsf{SNR})\right), \quad \forall i = 1, 2, \ldots K$, so that the achiveable sum-rate is $R_{achiev3}^{strong} = K\log(\mathsf{SNR})$. Comparing with R_{ub2}^{strong}, we see that

$$
\begin{aligned}
\delta_3^{strong} &= R_{ub2}^{strong} - R_{achiev3}^{strong} < K\\
&= O(K).
\end{aligned} \tag{35}
$$

Thus we see that, for the entire range of interference, there are signalling strategies that achieve a sum-rate within $O(K)$ bits of the sum-capacity, irrespective of the channel gains. ∎

V. Conclusions

In this work we have characterized to within a finite number of bits the sum-capacity of a K-user symmetric star Gaussian IC. The achievable schemes are designed from observations made in the corresponding deterministic case. Future research directions might include considering ICs with more connectivity than the star network, as also asymmetric parameter regimes, which would potentially increase our knowledge about capacity regions of various K-user Gaussian ICs.

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