Investing in Shared Spectrum

Chang Liu, Sihua Fu and Randall A. Berry
EECS Department, Northwestern University, Evanston, IL 60208
Email: {changliu2012,sihuafu2018}@u.northwestern.edu, rberry@eecs.northwestern.edu

***Abstract*—Sharing spectrum is a promising approach for expanding wireless access and increasing competition among wireless service providers. Indeed, this is a key motivation behind the recent regulations such as those for the 3.5 GHz band in the U.S. However, meeting this promise requires that service providers (SPs) have the incentives to invest in technology to be deployed in shared bands. This is not a forgone conclusion. Indeed by lowering entry barriers, sharing can promote more competition, but this also lowers revenue, making investment less attractive. In this paper, we study such scenarios for band of spectrum that is shared under a primary-secondary paradigm, by adopting a model developed by Liu and Berry in 2014. In their model, a primary SP and multiple secondary SPs compete for a common pool of customers using a shared band. In that work, any investments of the SPs was considered sunk, and it was shown that sharing improved both social welfare and consumer welfare over the case where the band was exclusively licensed to one SP. Here, we add an investment stage to this model, in which all of the SPs first decide on an investment level; given their investments, they again compete for customers. We characterize the sub-game perfect equilibrium of the resulting game and characterize the resulting consumer and social welfare. We show that a secondary SP needs a lower investment cost than a primary in order to enter the market. Moreover, at most one secondary SP will enter, even if multiple have low costs. Finally, we show that for large enough bandwidth, assigning the SP with the lower investment cost as the primary can provide more social welfare and consumer welfare than making it a secondary.**

I. Introduction

Sharing spectrum is a promising approach for meeting the rapidly growing demands for wireless access. In particular, the "primary-secondary" approach to sharing has received much interest. Under this approach, secondary spectrum users can share a band with a primary user, provided that the secondary users do not interfere with the primary's service. Such sharing techniques have been considered in several scenarios, e.g. [1], [2] and can be facilitated by either spectrum sensing [3], [4] or through the use of geolocation databases [5]. Indeed, such an approach underlies the policies adopted in the U.S. for the TV white spaces [6] and the 3.5GHz band [7]. A common feature of these policies is that secondary access is license exempt, e.g., this is true for the case of TV white-space devices and for the Generalized Authorized Access (GAA) tier in 3.5GHz. (For 3.5 GHz, there are three instead of two tiers of access, so we are abusing terminology somewhat by referring to these as secondary users.) One motive for this is that by not requiring an expensive license, it lowers the entry barriers and so may increase competition. However, this benefit is not a forgone conclusion, as the lower entry barrier also increases the risk that the spectrum becomes congested. The risk of such congestion can make service providers (SPs) reluctant to invest and offer service (e.g., see [8]). The focus of this paper is to gain insights into how these two effects can impact the economic benefits of primary-secondary sharing.

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We consider a model where there are both primary and secondary SPs that are competing to offer service to a common pool of customers. For example, this could model a scenario in the 3.5 GHz band, where the primary SP has a Priority Access License (PAL), while the secondary SPs are all GAA users. Note here we assume that Tier 1 federal incumbents are not present in the given area, while primary and secondary SPs are Tier 2 and Tier 3 users, respectively. We adapt the model for competition among such SPs established in [9], [10], which in turn uses models for price competitions among firms with congestible resources that have been studied in a number of different communities (see e.g., [11]–[13]); similar models have also been used to study competition among wireless SPs in other settings (e.g. [14]–[16]). In these models, SPs compete for customers by announcing a price for their services. Customers in turn select service providers based on a combination of their announced price and a congestion cost, which is increasing in the traffic seen by a provider. In the primary-secondary sharing model of [9], [10], the congestion cost of a primary SP depends only on its own traffic, while that of a secondary SP depends on the total traffic in the shared band. This captures the priority given to a primary user and also the "open" nature of secondary usage. In [9], [10], the focus was on modeling the resulting price competition among SPs. It was shown that, compared to the case where the spectrum was exclusively licensed, competition with this form of sharing always benefits consumers and also improved the overall economic welfare, which includes the profits of the SPs.

In [9], [10], the investment decision of the SPs was not modeled, i.e., any investment was considered sunk. Our main goal in this paper is to study the impact of investment in such settings. This is significant in part because in [9], it was shown the sharing does decrease the profit of the primary SP as well as cause the profits of the secondary SPs to be competed down to zero. Hence, it is not a forgone conclusion that SPs would invest in the shared band in the first place. To model investment, we consider a two-stage model in which first the SPs decide on a level of investment in the shared band and then, given the investment of each SP, they compete on prices for customers. The level of investment impacts the congestion that their customers experience for a given load but also reduces their profits. Similar models have been used previously in the literature such as in [17] to study technology choices of SPs and in [18], [19] to study investment and competition in unlicensed spectrum. As key difference of this paper from [**?**], [17], [18] is that in those papers tiered spectrum sharing was not considered. Related models have also been considered outside of the context of the wireless services market (e.g. [20] studied a similar model for firms in a generic congestible industry and [21] considered a model for cloud-based services).

Our main results are as follows. First we show that if a pure strategy Nash equilibrium exists, it will be one where at most a single secondary SP decides to invest, along with possibly, a single primary SP. Essentially, the threat of congestion keeps all secondary SPs but one from investing and so the benefits of increased competition due to secondary access being license exempt are limited. However, when both a primary and secondary SP invest, sharing does provide some level of increased competition. When all SPs face the same marginal costs of investment, this competition again leads to improvements in welfare over the case where a single SP has an exclusive license; however, the required cost level for a secondary SP to invest is much lower than that for a primary SP. We also compare this numerically to the case where the band is partitioned between two providers, with each having an exclusive license for its partition. Here, the comparison is more subtle, when the amount of bandwidth is small (or the investment cost is large) sharing generates the most consumer and social welfare; in this case it seems that sharing encourages the incumbent to invest more, which benefits both consumers and the overall welfare. However as the bandwidth increases (or the cost decreases), partitioning becomes better for consumer welfare, and then eventually for social welfare. In this case, as bandwidth increases, partitioning creates a more equal competition among the SPs, which in turn drives down prices and benefits consumers.

We also study a model where the primary and secondary SPs have different marginal costs of investment. In this case, we again characterize the investment choices of the SPs and show that it is even possible for only a secondary SP to decide to invest (if it has a very good costs), while the primary SP decides not to invest. Given two SPs with different costs, we also show that best welfare is obtained when the SP with the lower marginal cost is the primary (even though in this case the other SP may not invest). Interestingly, if the low cost SP is the secondary, it is even possible, that adding enough bandwidth so that the other SP will invest may lead to a loss in welfare.

The rest of this paper is organized as follows. We formally describe our model in Section II and present some preliminary results in Section III. We then analyze the model and compare the welfare obtained with spectrum partitioning and monopoly scenarios in Secion IV. Finally, we conclude in Section V.

II. Investment and Competition Model

A. A Two-stage Game

We consider a market in which a single primary SP (e.g., a PAL holder) is sharing a band of spectrum with a set of N secondary SPs (e.g., GAA users). Secondary access is "open" meaning that all secondaries have equal access to the spectrum, provided they do not interfere with the primary. We adopt a two-stage game model. In the first stage, the primary and all the secondary SPs simultaneously make investment decisions. The firms who do not invest will not serve any customers and so receive zero profits. The firms who invest more will provide better service. Given the investment choices of all SPs, in the second stage, the SPs then compete by announcing prices for their services. In the following we provide a more detailed description of this market.

1) Supply: The supply in this market is the service offered by the SPs. For each SP, this service is characterized by a congestion cost $g(y)$, where y represent the effective load that SP's customers experience. For simplicity, here we assume that $g(y) = \frac{y}{B}$, i.e., the congestion is proportional to the effective load (though in general this could be any increasing, convex function) and we view the parameter B as reflecting the bandwidth of the given band (so that more bandwidth means less congestion). We adopt the model from [9] to model the load experienced by the primary and secondary SPs. The load of the primary SP is given by $y_1 = x_1/I_1$, where x_1 is the mass of customers it serves and I_1 is its investment level. Hence, the primary's congestion cost will be increasing in the amount of customers it serves and decreasing in its investment. Note that the primary's congestion does not depend on the traffic of the secondaries, which capture the fact that the primary has strict priority in accessing this band. The load of the ith secondary is given by

$$y_i = \frac{x_1}{I_1} + \frac{\sum_{i=1}^{N} x_i^S}{I_i^S}$$

, where I_i^S is the investment of the ith secondary and x_i^S denotes the customer mass served by secondary SP i. Hence, each secondary SP not only suffers congestion from the primary but also from the other SPs' traffic. The primary traffic is included to model the secondaries' lower priority in accessing the channel. Note that if the primary invests more, this reduces the congestion seen by not only the primary but also by all secondaries. This models a case where increased primary investment enables it to serve its traffic faster and so make the channel available more often for secondary usage. Each secondary's investment only impacts its own load, but this load depends on the total traffic served by all secondaries. The reason for using the total traffic is to capture the "openness" of secondary access. This type of investment could model, e.g., investing in more cell sites, which would make the load (including the traffic of other SPs) per cell lower for that SP, but would not have a first order effect on the load seen by other SPs. Since all the secondaries' effective loads are the same, their service can only differ in their investment. In the competition stage, each SP announces its price; we denote by p_1 and p_i^S the prices for primary and secondaries, respectively.

2) Demand: Demand in the market comes from the customers requesting service. Here, we assume a single mass of infinitesimal customers and we normalize the total mass to be one. Each customer selects whether to get service and from which SP based on the *delivered price* defined as the sum of the congestion cost of the serving SP and its announced price. This models the fact that customers are sensitive to not just service price but also quality of service. The delivered price for the primary and secondaries are $p_1 + g(x_1/I_1)$ and $p_i^S + g(X/I_i^S)$, respectively, where X denotes the total traffic served by all the secondary SPs. Given a profile of announced prices in the competition stage, customers willing to accept service will select the SP which has the least delivered price (with ties broken randomly). The customers' demand is characterized by an inverse demand $P(x)$, which gives the delivered price at which x customers are willing to pay for service. Again to simplify our discussion, we assume a linear model where $P(x) = 1 - x$.

B. Equilibrium

To characterize an overall market equilibrium, we must specify the equilibrium assignment of customers to SPs and then the equilibrium price and investment decisions of the SPs.

1) Customer Equilibrium: Given the SPs' investments and prices, customers select the SP that offers the least delivered price. If that delivered price is higher than its willingness to pay, that customer will opt out of service. It follows that all SPs serving customers must have the same delivered price and this price must be no greater than $P(x)$, where x is the total customer mass served in the market. In other words, customers must be in a Wardrop Equilibrium [22]. Formally, a Wardrop Equilibrium can be specified by finding a constant K so that

$$p + \frac{x_1}{BI_1} \geq K, \quad \text{(w.e. if } x_1 > 0), \tag{1}$$

$$p_i^S + \frac{x_1}{BI_1} + \frac{X}{BI_i^S} \geq K, \quad \text{for } i \in N \text{ (w.e. if } x_i^S > 0), \tag{2}$$

$$1 - (x_1 + X) \leq K, \quad \text{(w.e. if } x_1 + X > 0). \tag{3}$$

(Here we use "w.e." to indicate "with equality".) Here, the solution K will be the delivered price in the market. The first two conditions guarantee that any provider serving customers will have the same delivered price; the last condition ensures that this delivered price is equal to $P(x_1 + X)$. An example of this for a market with 1 primary and 1 secondary SP is shown in Fig. 1. Using standard approaches it can be shown that such an equilibrium always exists and is unique (see e.g. [15]).

2) Provider Equilibrium: In the two-stage investment and competition game, SPs will first decide on an investment level and then compete to serve customers by announcing prices. Their profit is the difference between their revenue, given by the product of the announced price and mass of customers served, and the cost of their investment, given by product of the investment level, I, and the marginal cost of investment c. More specifically, the primary SP's profit f_1 is given by

$$f_1 = p_1 x_1 - c_1 I_1,$$

where c_1 is the marginal investment cost for the primary. The secondary SP i's profit is given by

$$f_i^S = p_i^S x_i^S - c_i^S I_i^S.$$

The marginal cost c_i captures the fact that SPs utilize their investment in different efficiencies. Note that in these expressions, the customers served are determined by the Wardrop equilibrium conditions in (1)-(3). In equilibrium, we assume that no SPs can increase its profit by changing its investment level or price. More precisely we study *sub-game perfect Nash equilibria*, in which given the investments of all SPs, all SPs choose prices so that no SP can improve its profit by unilaterally changing its price, while accounting for how customers will change to a new Wardrop equilibrium. Likewise, in the investment stage, no SP can unilaterally change its investment to improve its profit, while accounting for how this change will affect the pricing decisions of the other SPs. Hence, an overall market equilibrium is given by a sub-game perfect equilibrium of the SPs and the corresponding Wardrop equilibrium of the customers.

C. Welfare measures

Given such an equilibrium, the total *firm profit*, f is defined by the sum of the profits made by all SPs, i.e., $f = f_1 + \sum f_i^S$. The welfare of the xth customer served is the difference between the consumers' value for service, $P(x)$ and the delivered price they incur; customers that are not served receive zero welfare. The total *consumer welfare*, CW, is the integral of this over all customers; in Fig. 1, this is the area of the triangle formed by $P(x)$, the delivered price and the y-axis. The *social welfare*, SW, of the entire economy is the sum of the firm profit and the consumer welfare, i.e.,

$$SW = CW + f.$$

III. Preliminary Analysis

In this section, we give some preliminary results for our model. We start by characterizing the equilibrium for case where sharing is not allowed so that there is a single monopolist in the market. This will later be used as a benchmark. Then move on to sharing and give a basic property of the equilibrium with multiple secondaries in the market, which greatly simplifies our subsequent discussion. Finally, we also present a model for partitioning the spectrum between two providers, which is another benchmark we will use in numerical comparisons.

A. No Sharing

Without spectrum sharing, the primary SP serves as a monopolist and thus can use all the available bandwidth. It chooses its investment level and price to maximize its own profit. If it chooses not to invest its profit will be zero.

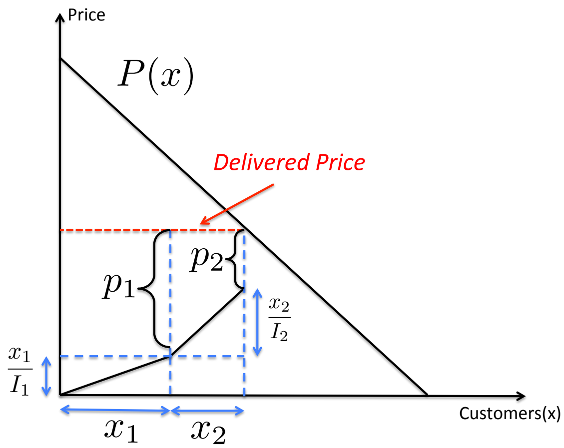

Fig. 1. Illustration of supply and demand for a model with a single primary SP and single secondary SP having a linear congestion cost and facing a linear inverse demand.

Otherwise, if it invests, it must serve a positive mass of customers to recover the investment cost. Assuming this case, since there is no competition, the SP's optimal investment and price are given by solving the following optimization problem:

$$\begin{aligned} \max_{p_1, I_1} \quad & p_1 x_1 - c_1 I_1 \\ \text{subject to} \quad & p_1 + \frac{x_1}{B I_1} = 1 - x_1. \end{aligned} \tag{4}$$

The constraint in this problem captures the customers being in a Wardrop equilibrium (since in this case only conditions (1) and (3) are present and we are assuming that $x_1 > 0$ so these conditions are met with equality). Solving this optimization gives $p_1^* = 1/2$ and $I_1^* = \frac{1}{2}\sqrt{\frac{1}{c_1 B}} - \frac{1}{B}$, which results in a positive profit when $c_1 \leq 1/4B$. In other words, provided that the primary SP is not facing too high of an investment cost, it will invest I_1^* and charge p_1^* for its service, resulting in a consumer welfare of

$$CW = \frac{1}{2}\left(\frac{1}{2} - \sqrt{\frac{c}{B}}\right)^2,$$

and a social welfare of

$$SW = \frac{1}{2}\left(\frac{1}{2} - \sqrt{\frac{c}{B}}\right) + CW.$$

Also note that if the primary invests, its optimal price does not depend on c or B; however, its investment is decreasing in both parameters. Also, the social and consumer welfare depend on c and B only through their ratio, i.e., a larger investment cost can be compensated for by a corresponding increase in bandwidth.

B. Sharing

Next we turn to the case where the spectrum is shared with one primary and $N \geq 1$ secondary SPs. In this case the primary and secondary SPs' investment and pricing decisions are coupled and so we must study the resulting sub-game equilibrium. Recall in this game, investment decisions are made first, followed by pricing decisions. We analyze this using backward induction, and so first characterize the pricing decisions of the SPs given the investment choices and then optimize the investments. Again all SPs have the option of choosing not to invest and achieving a pay-off of zero, in which case their pricing decision is irrelevant.

If the primary SP invests $I_1 > 0$, then its pricing decision is given by solving

$$\begin{aligned} \max_{p_1 \geq 0} \quad & p_1 x_1 - c_1 I_1 \\ \text{subject to} \quad & (1), (2), (3), \end{aligned} \tag{5}$$

where here we somewhat abuse notation and use (1), (2), (3) to refer to the fact that x_1 arises as a result of these Wardrop equilibrium conditions (which depend on the prices and investments of the other SPs). Note that in this stage the investment of the SP is sunk, and so the term $c_1 I_1$ does not impact the optimal choice of p_1. Also, note that the primary can always choose a price p small enough so that it draws a positive customer mass in the Wardrop equilibrium. It follows that in the pricing stage p_1 will be chosen so that (1) and (3) hold with equality.

Assuming the ith secondary SP invests $I_i^S > 0$, it seeks to solve

$$\begin{aligned} \max_{p_i^S \geq 0} \quad & p_i^S x_i^S - c_i^S I_i^S \\ \text{subject to} \quad & (1), (2), (3). \end{aligned} \tag{6}$$

Again, in this stage its investment is sunk. However, a secondary that invests $I_i^S > 0$ may not be able to serve any customers even if its price is zero. This arises due to the second Wardrop equilibrium conditions in (2) and the fact that the only difference in congestion costs among secondaries is in the investment cost I_i^s. For example suppose that another secondary SP j invests more than i and announces a price of p_j^S resulting in an equilibrium number of secondary customers X. Then if

$$p_j^S + \frac{X}{B I_i^S} < \frac{X}{B I_j^S}$$

it follows that no matter what price SP i announces, its delivered price will be larger than j's. In this case, the only solution to (6) is for x_i^S to be zero, yielding a loss of $-c_i I_i^s$.

We use the following result from [9] which greatly simplifies our analysis.

Lemma 3.1: In an equilibrium for the pricing sub-game, the announced price of at most one secondary that is serving traffic is positive.

To see why this is true, suppose at least 2 secondaries announce prices greater than 0 and are serving traffic. This means that the second Wardrop equilibrium condition (2) holds with equality for each of them. However, if either SP lowers its price slightly, then it will have a lower delivered price than the other secondary and thus draw all of the traffic from the other SP, which will increase its profit. It follows that these SPs would engage in a "price war" until at least one of their prices reach zero. If both SPs had invested the same amount, the outcome of this price war is that both of their prices are zero; otherwise, the SP with the smaller investment will end up with price of zero.

Using this result, we then have that at most one secondary SP will invest in the market

Lemma 3.2: In any sub-game perfect equilibrium, at most one secondary SP will invest.

This follows directly from the previous lemma, since if more than one secondary SP invested, the outcome of the pricing sub-game would be for one of the SPs that invested to announce a price of zero and so not make any profit to recover its investment. That SP would be better off changing its strategy to not investing.

Since it is only possible that one secondary SP serves the market, from now on, we only consider a single secondary SP and use a subscript 2 to indicate its parameters. For instance, the secondary's investment, price, customers served are I_2, p_2, and x_2, respectively. We focus in the next section on the interaction of this one secondary and the primary SP.

C. Partitioning

In order to compare the efficiencies of spectrum policies, we will also compare spectrum sharing with spectrum partitioning. Specifically, instead of sharing a band of spectrum with bandwidth B, we consider that the spectrum is partitioned into two parts with bandwidth αB and βB where $\alpha+\beta=1$. Each partition of the band is exclusively licensed to one of the SPs (no sharing). . Again, the SPs compete by first deciding on an investment level, but in this case it only affects the congestion in their partition. After deciding on investment levels, they compete by announcing prices, after which the users select SPs according to a Wardrop equilibrium.

Here since each SP is operating on its own band, they do not interfere with each other and thus their traffic does not affect the other SP's congestion. For instance, for the second SP, its effective price is $p_2 + \frac{x_2}{\beta B I_2}$ whereas in the sharing scenario, the secondary SP's effective price is $p_2 + \frac{x_1}{BI_1} + \frac{\sum x_2}{BI_2}$. Hence, in this case, the Wardrop equilibrium is specified by requiring a constraint of the form of (1) for each SP and no constrains like (2). Further, since they compete for the same pool of customers, they still encounter the same demand, i.e., if either is serving traffic it must be that the delivered price is equal to $P(x_1+x_2) = 1-(x_1+x_2)$.

To solve this game, we again use backward induction. We first find out the best responses of p_1 and p_2 for the two SPs as functions of I_1 and I_2 given the fact that they have observed each other's investment level. Then we come back to the investment stage and optimize the profit again in terms of I_1 and I_2 and find the optimal investment level. Note again, each SP has the option of not investing and earning zero profit.

Assuming both SPs invest, and serve traffic, then from the Wardrop equilibrium conditions we have

$$p_1 + \frac{x_1}{\alpha I_1} = 1-(x_1+x_2)$$
$$p_2 + \frac{x_2}{\beta I_2} = 1-(x_1+x_2).$$

To find the equilibrium in the price competition stage, we first use these equations to express x_1 and x_2 in terms of p_1 and p_2, giving

$$x_1(p_1,p_2) = \frac{(-\frac{1}{\alpha I_2}-1)p_1 + \frac{1}{\alpha I_2} + p_2}{\frac{1}{\alpha\beta I_1 I_2} + \frac{1}{\alpha I_1} + \frac{1}{\alpha I_2}}$$

and

$$x_2(p_1,p_2) = \frac{(-\frac{1}{\alpha I_1}-1)p_2 + \frac{1}{\alpha I_1} + p_1}{\frac{1}{\alpha\beta I_1 I_2} + \frac{1}{\alpha I_1} + \frac{1}{\alpha I_2}}.$$

Fixing p_2, SP 1's best response in the price competition is then given by solving

$$\begin{aligned} \max_{p_1\geq 0} \quad & p_1 x_1(p_1,p_2) - c_1 I_1 \\ \text{s.t.} \quad & p_1 + \frac{x_1(p_1,p_2)}{\alpha B I_1} = 1-(x_1(p_1,p_2)+x_2(p_1,p_2)). \end{aligned} \tag{7}$$

The other SP solves an analogous problem.

Using our expression for $x_1(p_1,p_2)$ in the objective function for the first SP, we have

$$f_1 = \frac{(-\frac{1}{\alpha I_1}-1)p_1^2 + (\frac{1}{\alpha I_2}+p_2)p_1}{\frac{1}{\alpha\beta I_1 I_2} + \frac{1}{\alpha I_1} + \frac{1}{\alpha I_2}}$$

which is concave in p_1 given p_2, I_1 and I_2 with p_1 in a compact set. Thus we obtain the optimal p_1 as

$$p_1 = \frac{\frac{1}{\alpha I_2} + p_2}{2(\frac{1}{\alpha I_2}+1)}.$$

We can do the same for the second SP and have the optimal p_2 as

$$p_2 = \frac{\frac{1}{\alpha I_1} + p_1}{2(\frac{1}{\alpha I_1}+1)}.$$

Given the best responses of the two firms, the equilibrium prices are given by

$$p_1 = \frac{2\alpha I_1 + \beta I_2 + 2}{3\alpha\beta I_1 I_2 + 4\alpha I_1 + 4\beta I_2 + 4},$$

and

$$p_2 = \frac{2\beta I_2 + \alpha I_1 + 2}{3\alpha\beta I_1 I_2 + 4\beta I_1 + 4\alpha I_2 + 4}.$$

To find the optimal investment, we put p_1 and p_2 back to the respective objective functions and find their best responses given the other firm's investment. Given the complexities of the problem, we use Matlab to find the numeric solutions for this stage of the game. Specifically, we iterate best response investments of the two firms until the investments converge.

It can be shown that when the available bandwidth is large enough, equally partitioning the spectrum, i.e., $\alpha=\beta=0.5$ will generate the most social welfare and consumer welfare. However, as bandwidth decreases, uneven partitioning can offer more social welfare and consumer welfare with both their maximum achieved at the extreme uneven partitioning - monopoly scenario. This will be illustrated in some of the numerical results in subsequent sections.

IV. Welfare Analysis

In this section, we study the equilibrium welfare obtained under spectrum sharing and compare that with the monopoly and spectrum partitioning scenarios. We divide our discussion into two cases when the primary and secondary SP have homogeneous and heterogeneous investment costs.

A. Homogeneous Investment Cost

We first state our main result in the following theorem and then provide a sketch of the proof and a discussion about the resulting equilibria.

Theorem 4.1: In a two-stage investment and competition game when the investment cost c for the primary and secondary are the same, there are three possible sub-game perfect equilibria, which depend on c and the available bandwidths B:

- When $c < \frac{B}{16}$, the primary will invest with I_1^*, where $I_1^* = \frac{2}{4+3BI_2}\sqrt{\frac{1+BI_2}{cB}} - \frac{1}{B}$ and the secondary will invest with I_2^* where I_2^* solves $4B - 3B^2 I_2 = c(4 + 3BI_2)$.
- When $\frac{B}{16} \leq c < \frac{B}{4}$, only the primary will invest with I_1^M where $I_1^M = \frac{1}{2}\sqrt{\frac{1}{Bc}} - \frac{1}{B}$.
- When $c \geq \frac{B}{4}$, then no one invests.

The full proof is given in Appendix V-A. Here we only sketch the main ideas. First we determine the equilibrium of the pricing sub-game by assuming that both SPs invest and serve customers. In this case, the best response for the primary SP is given by solving

$$\begin{aligned} \max_{p_1} \quad & p_1 x_1 - cI_1 \\ \text{subject to} \quad & p_1 + \frac{x_1}{BI_1} = 1 - (x_1 + x_2). \end{aligned} \tag{8}$$

Here, as in our analysis of the partitioning case, x_1 and x_2 are functions of p_1 and p_2, which can be determined via the Wardrop equilibrium constraints. Likewise, the secondary's best response is given by

$$\begin{aligned} \max_{p_2} \quad & p_2 x_2 - cI_2 \\ \text{subject to} \quad & p_2 + \frac{x_1}{BI_1} + \frac{x_2}{BI_2} = 1 - (x_1 + x_2). \end{aligned} \tag{9}$$

By solving these optimizations and finding the intersection of the best responses, we find the pricing equilibrium. Then we proceed to analyze the investment stage, where in the case only the primary invests, the corresponding pricing decision is the same as in the no-sharing scenerio in the previous section.

This result shows that there are equilibria in which both the primary and secondary invest; however, for the secondary SP to invest the cost must be much lower than for the primary to invest, namely the threshold cost value for the secondary is 1/4 of that of the primary. Note also that as B increases these thresholds increase in proportion to B, i.e., if more bandwidth is available, this can compensate for higher marginal costs.

Next we consider the asymptotic behavior as $B \to \infty$. For large enough B, eventually the seocndary will invest at a level I_2 that solves $4B - 3B^2 I_2 = c(4 + 3BI_2)$. From this, it can be seen that as $B \to \infty$, I_2 approaches 0. However, the product of B and I_2 approaches $\frac{4}{3}$. This is due to the fact that the original equation can be re-written as $4 - 3BI_2 = c(4 + 3BI_2)/B$, where the right-hand side goes to zero when B goes to infinity. In this case,

$$I_1 = \frac{2}{4 + 3BI_2}\sqrt{\frac{1 + BI_2}{cB}} - \frac{1}{B},$$

which also goes to zero as $B \to \infty$. Multiplying both sides of this equation by B, it can be shown that BI_1 approaches infinity. By putting all these into the social welfare, the limiting social welfare is given by 0.4477. For the monopoly scenario, when the bandwidth goes to infinity, it is easy to see the investment is zero while the monopolist charges a price equal to 1/2. Thus, the social welfare with a monopoly approaches 0.375. We summarize these in the following corollary.

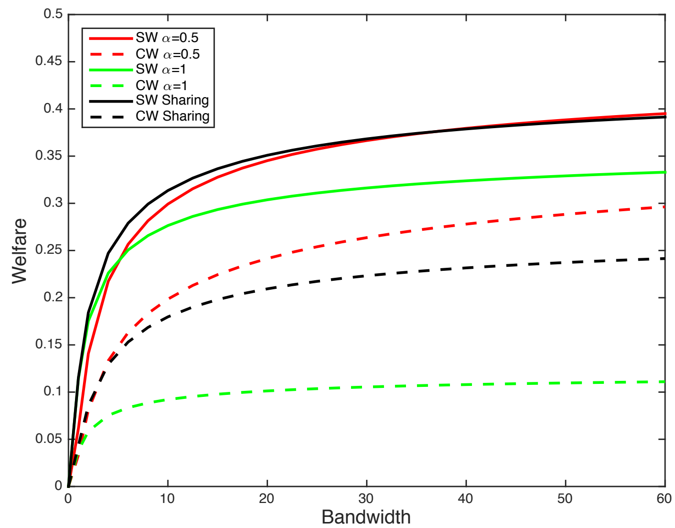

Fig. 2. Illustration of social welfare and consumer welfare when both the primary and secondary SPs' investment costs are the same ($c_1 = c_2 = 0.05$).

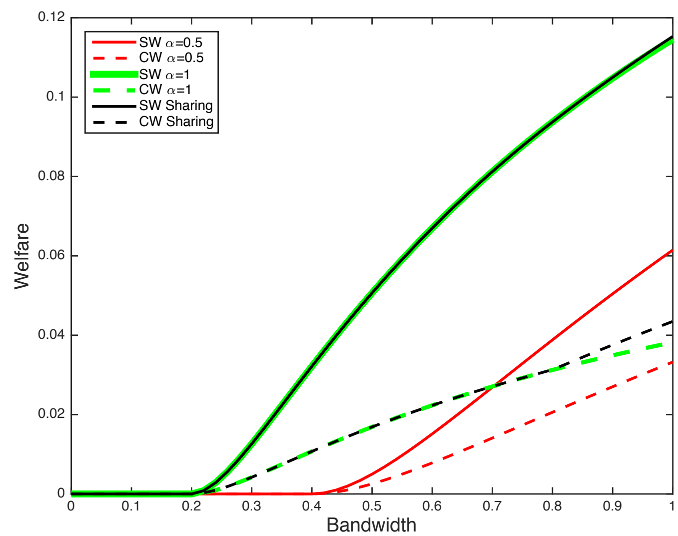

Fig. 3. Enlargement of Figure 2 when bandwidth is small.

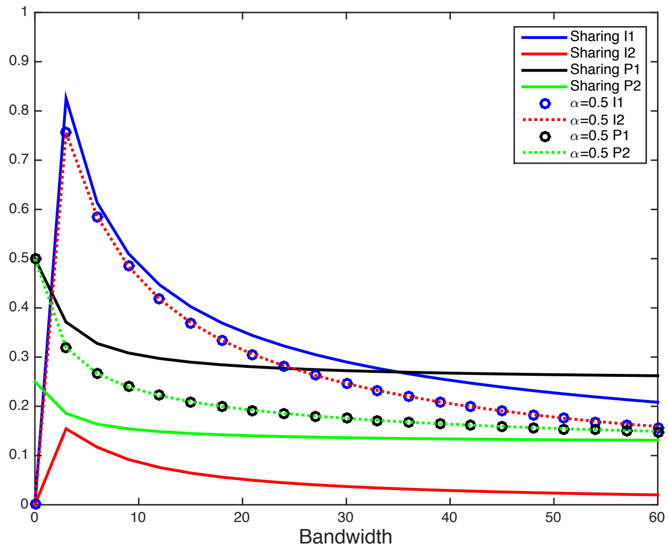

Fig. 4. Illustration of Social Welfare and Consumer Welfare when both primary and secondary SP's investment costs are the same $c_1 = c_2 = 0.05$.

Lemma 4.2: As $B \to \infty$, the product of bandwidth and investment for the primary and secondary approaches ∞ and $4/3$, respectively, i.e., $BI_1 \to \infty$ and $BI_2 \to \frac{4}{3}$.

Figures 2 and 3 show the equilibrium social welfare and consumer welfare as the bandwidth B varies. Here, three scenarios are considered including spectrum sharing, spectrum partitioning (α=0.5) and no sharing (α=1). This shows that as the bandwidth increases, the social welfare and consumer welfare increase in all three cases while the increasing rates are different. Figure 3 enlarges Figure 2. It shows that the primary starts investing at $B = 0.2$. When the bandwidth is between 0.2 and 0.8, the social welfare and consumer welfare for sharing and no sharing coincide since only one primary SP invests and acts as a monopolist in that regime. When $B = 0.8$ which is 16 times of the investment cost $c = 0.05$, the sharing and no sharing welfares start to differ since with sharing the secondary starts to invest which improves the social welfare and consumer welfare.

In Figure 2, sharing can provide most social welfare when the bandwidth is limited while evenly partitioning the spectrum exceeds that with large enough bandwidth though the difference is small. This is due to the fact that sharing makes the two SP invest more than partitioning while charging similar average prices as shown in Figure 4. However, the social welfare with partitioning is the lowest when only a small band of spectrum is available compared with both sharing and no-sharing. This is due to the fact that with limited resources, a monopoly can utilize most of the spectrum and having only one firm invest lowers investment costs compared to partitioning. When the bandwidth is large enough, partitioning allows the firms to not invest too much but benefit from the resource. Furthermore, the social benefits generated by spectrum sharing is always more than the no-sharing scenario. As we have shown earlier, as available bandwidth goes to infinity,

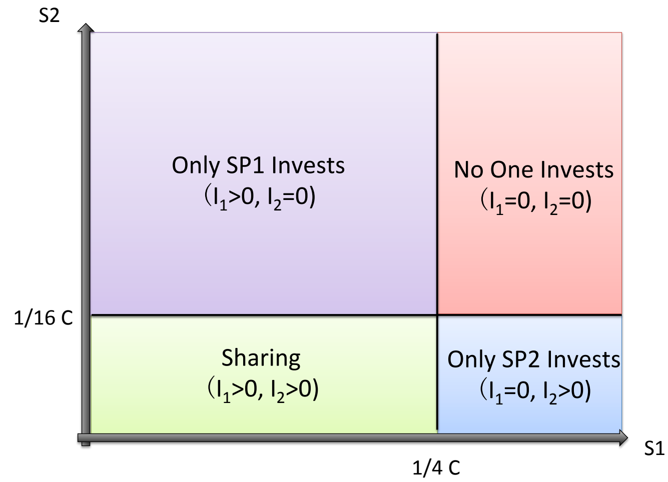

Fig. 5. Illustration of primary and secondary SP's investment profile based on their investment costs and total bandwidth.

the difference in social welfare with sharing and no-sharing approaches to 0.07. Sharing also achieves the largest consumer welfare with limited spectrum while partitioning bypasses sharing when the bandwidth becomes large. Again, sharing always creates more consumer welfare than no-sharing.

In [9] where investment is not modeled, sharing can always produce more social welfare and consumer welfare. With investment, we see the same trend, but the gains are lower, due to the fact that only one secondary SP invests.

More generally, it can be shown with a general convex increasing congestion cost and decreasing concave inverse demand, there are always three possible equilibria when the investment costs for both SPs are the same, that is

- No one invests,
- The primary invests and the secondary does not,
- Both the primary and secondary invest.

We leave a more detailed analysis of such a case to future work.

B. Heterogeneous Investment Cost

This section discusses the equilibrium when the marginal investment costs for the primary and the secondary are different, i.e., $c_1 \neq c_2$. As we will see later, there are still three possible cases as before. However, now when there is only one SP investing, it may not be the primary. We first state our main result below.

Theorem 4.3: For our two-stage investment and competition game, there are four possible sub-game perfect equilibria with linear congestion costs and inverse demand depending on the investment costs c_1, c_2 and bandwidth B.

- When $c_1 > B/4$ and $c_2 \leq B/16$, then only the primary invests with $I_1^M = \frac{1}{2}\sqrt{\frac{1}{Bc_1}} - \frac{1}{B}$.
- When $c_1 \leq B/4$ and $c_2 > B/16$, then only the secondary invests with $I_2^M = \frac{1}{2}\sqrt{\frac{1}{Bc_2}} - \frac{1}{B}$.

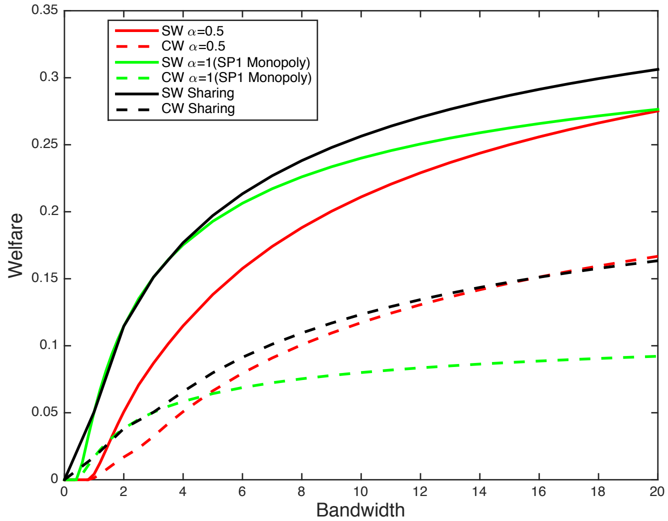

Fig. 6. Welfare comparisons among spectrum sharing, partitioning and monopoly scenarios when $c_1 = 0.1$ and $c_2 = 0.2$.

- When $c_1 \leq B/4$ and $c_2 \leq B/16$, then both SPs invests with $I_1^* = \frac{2}{4+3BI_2}\sqrt{\frac{1+BI_2}{c_1 B}} - \frac{1}{B}$ and the secondary will invest with I_2^* where I_2^* solves $4B - 3B^2 I_2 = c_2(4 + 3BI_2)$.
- When $c_1 \leq B/4$ and $c_2 \leq B/16$, then no SP invests .

The proof is similar to the case when the investment costs are the same so we omit it here. We show this investment profile in Figure 5 with a given bandwidth. This shows that it has a lower barrier for the primary SP to invest than the secondary since the primary simply needs to have a investment cost lower then 1/4 of the bandwidth while it needs to be 1/16 for the secondary to invest.

An example of the resulting equilibrium investment and price levels as well as welfares are shown in Fig. 6 and Fig. 7. These shows that when $B = 0.4$, the primary starts to invest and when $B = 3.2$ the secondary starts to invest. As the bandwidth grows larger, both of them have to decrease their prices in order to gain more customers. Again the social welfare and consumer welfare for sharing are the same as in the no-sharing case, when $B < 3.2$ but larger otherwise. We can see that partitioning does not provide as much social welfare and consumer welfare as sharing with limited bandwidth. But as bandwidth grows, this difference starts to shrink.

There exists another case when the marginal investment costs for the secondary is very small, then it is possible for only the secondary SP to invest, in which case it will act as a monopolist. This case can cause a dramatic change in social welfare compared to the other cases as shown in Fig. 8. This is because when the bandwidth is large enough for the secondary to invest, i.e., $B > 16c_2$, but small enough to prevent the primary from investing, i.e., $B < 4c_1$, then only the secondary will invest and act as a monopolist in the market. This corresponds to the area when $0.32 < B < 0.8$ in Fig. 8. At this region, the social welfare and consumer welfare are the

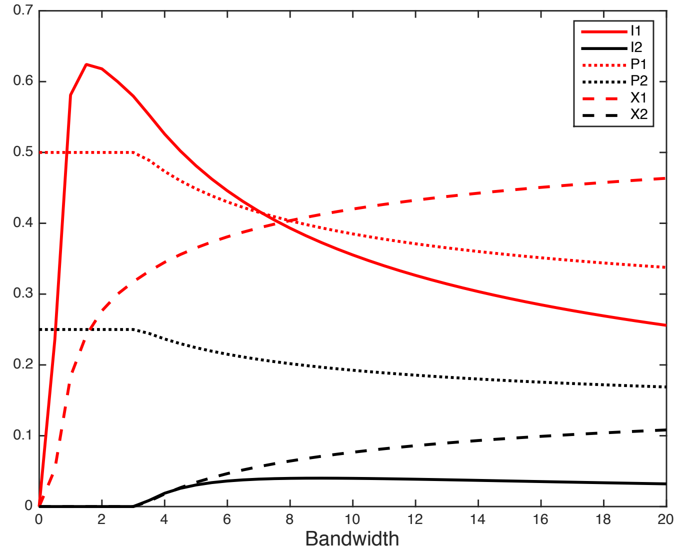

Fig. 7. Investment levels, announced prices and serving customers for the primary and secondary SPs when $c_1 = 0.1$ and $c_2 = 0.2$.

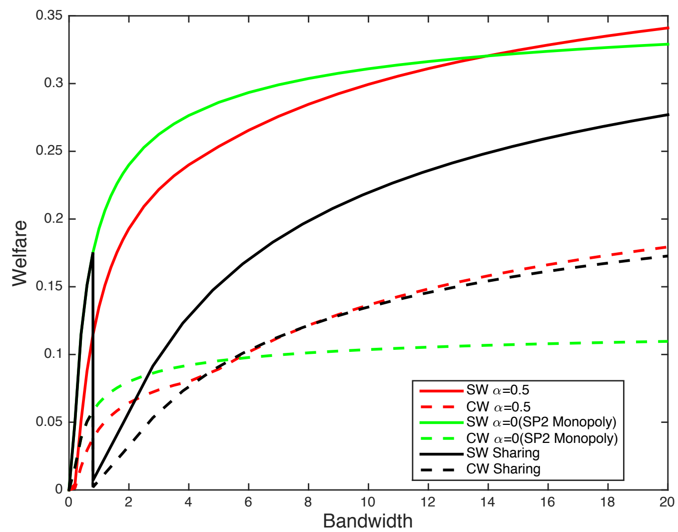

Fig. 8. Welfare comparisons among spectrum sharing, partitioning and monopoly scenarios when $c_1 = 0.2$ and $c_2 = 0.02$.

same as if the secondary was a monopolist. However, when the bandwidth grows larger, the primary will invest. Since it has a larger investment cost and less incentive to invest, it will invest less which makes the secondary's customers suffer from the additional congestion coming from the primary. Hence, at $B = 4c_1 = 0.8$, the social welfare has a significant drop due to the entry of the primary and then slowly increases as the bandwidth grows. Compared with the case when the primary has less investment cost as in Fig. 7, this big drop in social welfare indicates the fact that having a primary SP with a high investment cost may create counter-intuitive outcomes, i.e., the social welfare may drop with increasing bandwidth.

Indeed, Fig. 9 shows the contrast in social welfare when the two distinct SPs switch their roles as primary and secondary

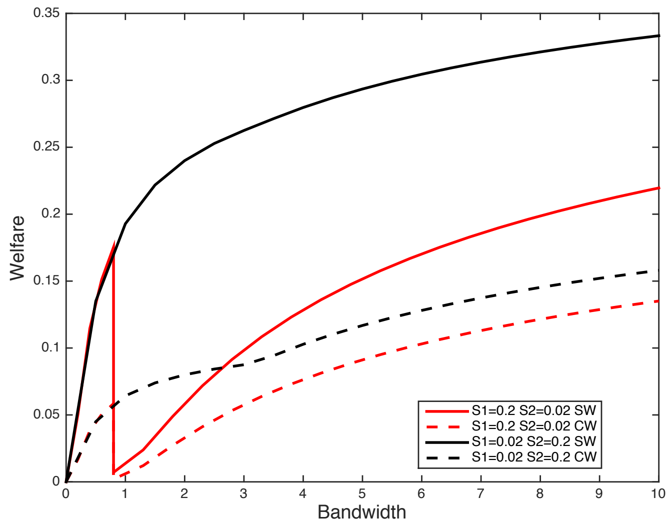

Fig. 9. Welfare comparisons when switching the investment costs of the primary and secondary SPs.

SPs. It shows that when the primary has a lower investment cost and the secondary has a higher investment cost, i.e., $c_1 = 0.02$ and $c_2 = 0.2$, the primary will invest and act as a monopolist when the bandwidth is small and then when $B = 16c_2 = 3.2$, the secondary decides to enter and compete with the primary. This has a influence on the social welfare and consumer welfare but does not affect its trend in growing. However, when exchanging the investment cost of the primary and secondary, the primary will enter at $B = 4c_1 = 0.8$, where again a drop in welfares happens. And as shown in the figure, this setting has much lower social welfare and consumer welfares. This also indicates that the assignment of which SP is primary or secondary can be important. As a policy planner, if the spectrum is to be shared, it would be better in terms of social welfare to make the SP with the lower investment costs, the primary.

V. Conclusion

This paper studied a two-stage game for investment and competition when spectrum is shared between a primary and secondary SPs. We have shown that for the given model, only one secondary SP will invest and compete with the primary. Further, the entry barriers for the primary and secondary SPs are different. With a linear model, only when the bandwidth is larger than 4 times its investment cost, i.e., $B > 4c_1$, will the primary invest. In contrast, the available bandwidth need to be 16 times a secondary's investment cost for it to enter. The welfare analysis also indicates that putting the SP with the lower investment costs as the primary can offer larger social welfare and consumer welfare with larger bandwidth. This may provide some insights into how spectrum rights can be allocated.

The model we studied was stylized and could be made more realistic in a number of ways such as by using other models for the congestion costs. For example, in [23] a more refined model for sharing was studied based on models of priority queues. Here we assumed that all users weighed congestion and price in the same way; richer models that allow for different classes of users could be considered. We assumed that Tier 1 users were not present; introducing them into the model is another potential future direction. Also, more detailed models for investment could be developed that account for different ways that SPs might invest, e.g. deploying smaller cells or different technologies.

Appendix

A. Proof of Theorem 4.1

Proof: To solve the optimization problem, we use backward induction: first we find the optimal price for the primary and the secondary given the other's price and fixed investment levels. Then we find the optimal investment levels.

First, consider at the competition stage when the investment levels for both SPs are sunk. Specifically we treat I_1 and I_2 as given at this point and solve the price competition game between the primary and secondary. From the first constraint from both SPs, we can obtain

$$x_2 = BI_2(p_1 - p_2). \tag{10}$$

Putting this back to the objective function of the secondary, its profit turns out to be $f_2 = -BI_2p_2^2 + BI_2p_1p_2 - cI_2$. Given the announced price from the primary, this objective is a concave function with its variable in a compact set. Thus a unique solution is obtained by taking the first derivative. The optimal price for the secondary is

$$p_2 = \frac{1}{2}p_1. \tag{11}$$

Now we turn to the primary. From (10) and the first constraint in (8), x_1 can be represented by

$$x_1 = \frac{-(1 + BI_2)p_1 + 1 + BI_2p_2}{1 + 1/(BI_2)}.$$

By plugging in x_1 to the objective of the primary, it is reduced to $f_1 = \frac{-(1+BI_2)p_1^2+(1_BI_2p_2)p_1}{1+1/(BI_2)} - cI_1$. Again this can be solved by convex optimization giving the unique solution

$$p_1 = \frac{2}{4 + 3BI_2}. \tag{12}$$

From the best responses of the primary and secondary given each other's price, (11) and (12), we have the following equilibrium at the price competition stage:

$$\begin{aligned} p_1 &= \frac{2}{4 + 3BI_2}, \\ p_2 &= \frac{1}{4 + 3BI_2}. \end{aligned} \tag{13}$$

Second, the optimal investments for both SPs can be obtained by putting the optimal prices back to each SP's objective function with the other's investment profile given. For the secondary, its profit reduces to

$$f_2(I_2) = \frac{BI_2}{4 + 3BI_2} - cI_2.$$

Here the profit of the secondary does not depend on the primary's investment. So to optimize its profit, the secondary only needs to find its own optimal investment. The first derivative and the second derivative of the secondary's profit are given by $f_2^{'}(I_2) = \frac{B}{(4+3BI_2)^2} - \frac{6B^2 I_2}{(4+3BI_2)^3} - c$ and $f_2^{''}(I_2) = \frac{6B^2(3BI_2-8)}{(4+3BI_2)^4}$. The second derivative of the objective function is not always negative. When $I_2 = 0$, this yields $f_2^{''}(I_2) < 0$. And as I_2 increases, $f_2^{''}(I_2)$ increases. In addition, it can be shown that as I_2 goes to infinity, $f_2^{''}(I_2)$ is negative. This means the first derivation of the secondary's profit first decreases and then increases but when I_2 goes to infinity, $f_2^{'}$ is negative. Further, since $f_2(0) = 0$, if the first derivative at 0 is negative, then the secondary can not gain positive profit thus will opt not to invest. In other words, only when $f_2^{'}(0)$ is positive, the secondary SP can make an investment and obtain a positive profit. Thus to make $I_2 > 0$, it must be that $f_2^{'}(0) > 0$, i.e., $c < \frac{B}{16}$. And when it invests, it finds the optimal profit by setting the first derivative to zero which yields that I_2 solves

$$4B - 3B^2 I_2 = c(4 + 3BI_2)^3.$$

Now we turn to the optimization problem for the primary SP. By plugging in the optimized prices, the objective for the primary becomes to maximize

$$f_1(I_1) = \frac{4(BI_2+1)}{(4+3BI_2)^2} - \frac{4(BI_2+1)}{(4_3BI_2)^2}\frac{1}{(BI_1+1)} - cI_1.$$

With the investment level of the secondary given, the maximization problem above can be transformed to minimize $L = \frac{4(BI_2+1)}{(4_3BI_2)^2}\frac{1}{BI_1+1} + cI_1$ which can be further reduced to minimize

$$\begin{aligned} L &= \frac{4(BI_2+1)}{(4_3BI_2)^2}\frac{1}{(BI_1+1)} + cI_1 \\ &= \frac{4(BI_2+1)}{B(4_3BI_2)^2}\frac{1}{(I_1+1/B)} + c(I_1+1/B) - c/B \\ &\geq 2\sqrt{\frac{4c(cI_2+1)}{B(4+3BI_2)^2}} - c/B. \end{aligned} \tag{14}$$

The last inequality comes from the fact that $a/x + bx \leq 2\sqrt{ab}$ with equality at $x = \sqrt{a/b}$. Thus the minimum of L is achieved at

$$I_1 = \frac{2}{4+3BI_2}\sqrt{\frac{BI_2+1}{Bc}} - \frac{1}{B},$$

with the constraint that $I_1 \geq 0$. When the secondary does not invest, $I_1 = \frac{1}{2}\sqrt{\frac{1}{Bc}} - \frac{1}{B}$. So in order for the primary to invest in the first stage, i.e., $I_1 > 0$, it must be that $c < B/4$. This completes the proof.

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