Pricing, Bandwidth Allocation and Service Competition in Heterogeneous Wireless Networks

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***Abstract*—Small-cells deployed in licensed spectrum can expand wireless service to low mobility users, which potentially reduces the demand for macro-cellular networks with wide-area coverage. Introducing such heterogeneity also makes network resource allocation more complicated. To understand these challenges and tradeoffs we present a two-tier heterogeneous wireless network model with two types of users: mobile users that can only connect to macro-cells; and fixed users that can associate with either macro-cells or small-cells. We study pricing strategies and bandwidth allocation across macro- and small-cells, assuming both monopoly and competitive Service Providers (SPs). For a monopoly SP, we characterize the revenue-maximizing prices and bandwidth allocations. We then consider a competitive scenario, and we show the existence of a unique Nash equilibrium. The possible Nash equilibria for different system parameters are sorted into four categories corresponding to whether or not different SPs assign bandwidth to the macro- and/or small-cells. We also study the allocations that maximize social welfare. For the competitive scenario, we characterize the conditions under which the optimal social welfare is obtained in equilibria as the number of SPs tends to infinity. Case study examples and numerical results illustrate the corresponding pricing and bandwidth allocations.**

***Index Terms*—HetNet, pricing, bandwidth allocation, service competition, game theory, network economics.**

I. INTRODUCTION

One of the key approaches the cellular industry is taking to accommodate the accelerating increase in wireless data demand is to shrink cell sizes. This is leading to the deployment of heterogeneous networks (HetNets) containing different types of base stations/access points with different transmission powers, coverage ranges, and data rates [3]–[6]. At the same time, the user population is also becoming more heterogeneous, with different mobility patterns and service demands. These trends will likely accelerate as 5G networks are deployed [7].

A substantial amount of research on HetNets has focused on interference management and related issues, such as coverage and rate estimation [8], [9], user association [10], [11], load balancing [12]–[14], carrier aggregation [15], massive MIMO [16], and cell cooperation [17]. This body of work applies to a network with given load, or traffic pattern, and does not consider how that load may depend on strategic decisions on the part of the service providers (SPs). These strategic decisions include pricing plans that can discriminate among different types of services and can compete with similar plans offered by other SPs. These pricing strategies are coupled with decisions on how to allocate resources such as bandwidth across cells and different user types, which provide another dimension in which SPs may compete.

In this paper, we study joint pricing and bandwidth optimization of SPs with both heterogeneous infrastructure and users. Both a single monopolist SP and competing SPs are considered. Each SP's HetNet consists of two types of cells (macro- and small-) and two types of users, mobile and fixed. The small-cells serve fixed users only, whereas the macro-cells can serve either user type. Associated with each cell type is its total rate capacity, where the total rate capacity of a small-cell is typically larger than that of a macro-cell with the same amount of bandwidth [18]. The SP announces two prices for accessing its macro- and small-cells, respectively, and users then choose an association across SPs and cell types that maximizes their utility minus access cost. We assume each SP has a fixed allocation of bandwidth and characterize the optimal split of bandwidth across macro/small-cells.

A. Contributions

We summarize our main results. The pricing and bandwidth allocation decisions are obtained via a two-stage process. In the first stage each SP determines how bandwidth is split between macro- and small-cells. In the second stage, each SP sets two prices for accessing the macro- and small-cell networks. This order is motivated by the observation that determining the bandwidth split may occur over a slower time-scale than price adjustments. We analyze the joint pricing and bandwidth allocation decisions by first characterizing the pricing strategies given a fixed bandwidth allocation for all SPs. We then determine the bandwidth allocation based on the pricing results obtained in the first step. The following results are obtained:

1. *HetNet market structure:* At equilibrium the macro-cells only serve mobile users, and fixed users only associate with small-cells. The prices in macro-cells are always higher than the prices in small-cells. This market structure applies irrespective of the number of SPs, and whether the SP(s) maximize

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(individual) revenue or social welfare. This is consistent with the current small-cell deployments in practice, where small-cells are primarily used in indoor systems [19] [20].

2. *Bandwidth allocation:* In the monopoly scenario (single SP), subject to some conditions on user utility functions, there always exists a unique optimal pricing and bandwidth allocation scheme. In the competitive scenario (multiple SPs), there is always a unique subgame-perfect Nash equilibrium. Furthermore, the equilibrium falls into one of four classes, corresponding to whether or not subsets of SPs allocate bandwidth to macro- or small-cells only, or to both.

Additional properties of the equilibrium are also characterized, for example, it cannot be the case that one SP provides only macro-cell service while another provides only small-cell service. (However, one SP can provide macro-cell service and another can offer both macro- and small-cell service.) Moreover, we show that the Nash equilibrium can be computed via a sequence of coordinate gradient-based updates, and use this to illustrate numerically how the equilibrium changes with initial bandwidth endowments.

3. *Social Welfare:* In addition to considering individual revenue maximization by the SPs, we also characterize the prices and bandwidth allocation that maximize social welfare, namely, the sum utility of all users in the network. In the monopoly scenario, we show that the optimal pricing rule for social welfare maximization is the same as for revenue maximization. However, the optimal bandwidth allocation is generally different for the two objectives.

4. *Competitive limit:* We investigate the asymptotic setting as the number of SPs increases to infinity, and show that efficiency (maximum social welfare) is achieved only for two of the equilibrium categories: where all SPs allocate bandwidth to both macro- and small-cells, or to small-cells only. This is similar to models of Cournot competition in which as the number of firms tend to infinity, perfect competition emerges resulting in optimal social welfare [21]. Here this occurs when the SPs compete with each other in both macro- and small-cells or in small-cells only. In contrast, if competition is absent or incomplete in either macro- or small-cells, even if we have infinite number of SPs, the equilibrium generally is not socially optimal.

B. Related Work

There is an extensive literature on pricing and resource allocation in wireless networks (see [22] for a comprehensive survey). Models for studying these issues in HetNets can be roughly divided into two categories. In the first category, small-cells are considered as an enhancement to macro-cells [23]–[25], as opposed to a separate service as in our work. The work in [23] investigates the interplay of interference and service pricing on user adoption of small-cells when small- and macro-cells operate in common spectrum and when they operate in fixed separate bands. The authors conclude that almost all users choose small-plus-macro service and pay a higher price. These results are extended to an arbitrary number of services using a Stackelberg game formulation in [24], where the bandwidth allocation is also optimized and numerically studied. It is observed in [24] that some minimal bandwidth must be allocated to small-cells in order to provide enhanced services, but at higher prices. In [25], a sequential game model with only optimal pricing is proposed, with the conclusion that it is beneficial to make all small-cells open to guest users with only macro-cell service. In contrast, in [26] the authors study optimal spectrum allocation in HetNets across licensed and unlicensed spectrum assuming centralized control, and without service pricing considerations. Moreover, all these works only consider a single SP without any service competition.

In contrast, the second category assumes macro-cell and small-cell services are separate services as in our work. When considering both pricing and bandwidth allocation decisions for a SP, there are two methods commonly used in the literature. The first is using a sequential decisions where pricing and bandwidth allocation are separated into two stages as adopted in this paper. Examples of this include [27]–[29]. The authors in [27] assume that bandwidth is allocated prior to pricing decisions. The results show that combining pricing and spectrum allocation increases the SP revenue and motivates users to adopt small-cell service. This approach is extended to different small-cell deployment types in [28].

The Shannon rate objective is used in [29] in a two-stage Stackelberg game model among an operator and users. In Stage I, the operator determines the price charged to users, whereas in Stage II, each user decides how much bandwidth it requests. It is assumed that users always achieve the best service by connecting to small-cells leading to the conclusion that all available bandwidth should be allocated to small-cells. However, in our paper we find that in some equilibria macro-cells should always get some bandwidth.

Instead of a sequential model, the authors in [24] optimize pricing and bandwidth allocation jointly. They show for any fixed pricing and fixed bandwidth allocation policy, there exists at least one equilibrium. However, since multiple equilibria might exist, the operator is assumed to choose either the worst or the best equilibrium, which are compared numerically. Note that the sequential process used in our paper leads to a unique equilibrium.

While these models only focus on a single SP, there has also been work that analyzes competition among multiple SPs and its impact on pricing and bandwidth. Reference [30] studies service competition and pricing strategies with fixed bandwidth allocations, and presents a comprehensive survey of the issues related to pricing in HetNets and possible approaches to solving the pricing problem. In [31], an incentive mechanism is designed in which a macro-cell SP can pay small cell SPs. Dynamic pricing and open access are shown to be better than static strategies. In [32], the authors consider two types of connections, premium and best-effort and develop a competitive pricing model for best-effort connections. The preceding work focuses on a Bertrand game in which the SPs compete on price only, whereas our paper also optimizes the bandwidth allocation.

Optimization of spectrum resources with multiple SPs is considered in [33], [34]. However, there the SPs compete to acquire spectrum, as opposed to optimizing bandwidth over different cell types. Pricing and bandwidth allocation in other settings has also been studied including SPs sharing unlicensed spectrum [35], and dynamic spectrum sharing [36].

The remainder of the paper is organized as follows. The system model is introduced in Section II. Optimal pricing and bandwidth allocation are presented in Sections III and IV, respectively. Social welfare is studied in Section V. We present some examples and numerical results in Section VI and conclude in Section VII. All proofs can be found in the appendices.

II. System Model

A. Network Model

We consider the downlink of a cellular network containing macro- and small-cells. Macro-cells correspond to transmitters with high transmission power, and therefore large coverage range, such as cellular base stations. In contrast, small-cells consist of transmitters with low transmission power, and therefore local coverage range, such as femto- or pico-cells. The users are also heterogeneous: mobile users are assumed to have high mobility, and fixed users are relatively stationary. In this setting, fixed users can associate with either macro- or small-cells (but not both) whereas mobile users can only be served by a macro-cell. This models the case where mobile users are unable to connect to a small-cell as they would move out of coverage rapidly, or the case where small-cells provide only indoor coverage.

The macro-cells and small-cells are assumed to be uniformly deployed over a given area. We normalize the density of macro-cells to one, and assume that the users are non-atomic with the densities of mobile users and fixed users denoted as N_m and N_f, respectively. Note that the heterogeneity of the users can also arise from an equivalent model that assumes $(N_m + N_f)$ as the total density of users who are mobile with probability $N_m/(N_m + N_f)$ and stationary with probability $N_f/(N_m + N_f)$.

We assume N SPs that operate different networks, with the same density of macro- and small-cells across the given region. This homogeneous cell density assumption greatly simplifies our analysis.[1] Denote the set of SPs as $\mathcal{N}$. When $N = 1$, this corresponds to the monopoly scenario, whereas $N \geq 2$ corresponds to a competitive scenario. Each SP i has its own exclusively licensed band of spectrum with bandwidth B_i, where B_i may vary across different SPs.[2] We assume all macro-cells and small-cells use separate bands, therefore each SP i decides how to split its bandwidth B_i into $B_{i,M}$, the bandwidth allocated to macro-cells, and $B_{i,S}$, the bandwidth allocated to small-cells. [3]

We assume that the macro-cells for SP i can provide a total (average) data rate of $C_{i,M} = B_{i,M}R_0$, where R_0 is the (average) spectral efficiency of the macro-cells.[4] In practice, R_0 depends on the transmission parameters and the inter-site spacing. The total available rate in small-cells for SP i is then $C_{i,S} = \lambda_S B_{i,S} R_0$, where $\lambda_S > 1$ reflects the increase in spectral efficiency due to smaller cell size, and possibly greater deployment density. The quantities $C_{i,M}$ and $C_{i,S}$ can then be interpreted as the service capacities of the macro- and small-cells, respectively, for SP i. We will denote by $K_{i,M}$ and $K_{i,S}$ the mass of users connected to the macro- and small-cells of SP i, respectively. (Note that $K_{i,S}$ consists of fixed users only, whereas $K_{i,M}$ can consist of both mobile and fixed users.)

B. Market Model

Each SP offers two distinct types of service: macro-cell service and small-cell service. SP i sets a price *per unit rate* for users associating with macro-cells or small-cells, namely, $p_{i,M}$ and $p_{i,S}$. The SPs do not further distinguish between different users subscribing to the cells. Since we allow fixed users to access either type of cell, the average price for fixed users can be interpreted as the average of the macro/small-cell prices weighted by the fraction connected to each. This allows for a form of differentiated pricing between the two user types.

The users are assumed to be homogeneous in that each user is endowed with the same utility function $u(r)$ depending on its received service rate r. We make the following assumptions on the class of utility functions studied.

Assumption 1: $u(r)$ has the following properties:

a) $u(0) = 0$ so that zero utility is received for no service;
b) $u(r)$ is strictly increasing and concave with r, i.e., users have "elastic" rate requirements;
c) $ru'(r)$ is twice differentiable, strictly increasing and concave for $r \geq 0$;
d) $u'(r) < +\infty$ for $r > 0$; and
e) $u''(r)$ is increasing for $r \geq 0$.

Properties (a) and (b) are common; properties (c)-(e) are more restrictive and made to facilitate our analysis. These are satisfied by many typical utility functions used in the literature such as $\alpha-$fair functions $u(r) = \frac{r^{1-\alpha}}{1-\alpha}, \alpha \in (0, 1)$, and $u(r) = \log(1+r)$. Note that $\alpha-$fair functions approach linear as $\alpha \to 0$, and log as $\alpha \to 1$, even though neither the linear nor the log utility function satisfies Assumption 1.

C. User Association

Each user chooses an SP, and fixed users also choose between macro- or small-cell service. Given a set of prices across SPs, each user makes this decision by maximizing its net payoff W. For a given service with price p, this is given by:

$$W = \max_{r \geq 0} \quad u(r) - pr, \tag{1}$$

where the optimal r is the rate a user would select if using this service. Under Assumption 1, (1) results in all users having a unique rate demand function given by $D(p) =$

[1] We can relax this assumption and consider different densities of small-cells for SPs, see for example [37] where these densities are determined by each SP's level of investment.

[2] For the monopoly SP scenario, we will ignore the subscript.

[3] In this paper we only consider licensed spectrum. We can relax this assumption and consider introducing unlicensed spectrum, see for example [38] where the impact of unlicensed spectrum is analyzed.

[4] We assume that these quantities represent the average value over the time-scale at which users select services.

$\max((u')^{-1}(p), 0)$. The maximum net payoff W^* for a user from a service with price p is then

$$W^* = u(D(p)) - p(D(p)). \tag{2}$$

It is easy to verify that W^* decreases with p and $W^* > 0$ as long as $D(p) > 0$. Therefore, if there is enough bandwidth to satisfy the rate demand, users choose the SP with the lowest price. That is, mobile users choose the SP that offers the lowest macro-cell price, while fixed users connect to either a macro- or small-cell with the lowest price. However, in general the total rate demand may exceed the supply in a macro- or small-cell. We then assume the following *user association rule*[5]:

- Suppose all prices $p_{i,M}$ and $p_{i,S}$ are different. We then assign the mobile users to macro-cells by filling up the available capacity starting with the SP i with lowest $p_{i,M}$, then the next lowest, and so forth until either all users are served or there is no more capacity.
- Similarly, for the fixed users we fill up the available capacity in order of increasing prices. Here, however, fixed users can choose to associate with the small-cells, or macro-cells with any remaining capacity after serving the mobile users. Moreover, mobile users have priority over the fixed users when connecting to a macro-cell.
- If there are SPs with prices that are the same, then those corresponding cells are filled simultaneously, and we allocate the users across cells in proportion to the cell capacities. Once a particular cell's capacity is exhausted, then the leftover demand continues to fill the remaining cells in a similar fashion.

This association rule leads to a unique assignment of users for any given set of prices and bandwidths. Suppose we change this rule, for example, so that mobile users do not have priority for cells. Then the assignment will not be unique as macro-cells can contain an arbitrary mix of mobile and fixed users if they have the same price as small-cells, which makes the equilibrium analysis more complex.

D. SP Optimization

Each SP i chooses its bandwidth partition ($B_{i,M}$, $B_{i,S}$) and prices ($p_{i,M}$, $p_{i,S}$) to maximize its revenue, which is equal to the aggregate amount paid by all users connecting to its macro-cells and small-cells:

$$\underset{\substack{B_{i,M}, B_{i,S} \\ p_{i,M}, p_{i,S}}}{\text{maximize}} \quad S_i = p_{i,M} K_{i,M} D(p_{i,M}) + p_{i,S} K_{i,S} D(p_{i,S}), \tag{3a}$$

$$\text{subject to} \quad B_{i,M} + B_{i,S} \le B_i, \tag{3b}$$

$$B_{i,M}, B_{i,S} \ge 0, \tag{3c}$$

$$0 < p_{i,M}, p_{i,S} < \infty. \tag{3d}$$

Note that $K_{i,M}$ and $K_{i,S}$ depend on the actions of the other SP's through the association rule described in Section II-C.

Alternatively, a social planner, such as the FCC, may seek to allocate bandwidth and set prices to maximize social welfare, which is the sum utility of all users. This can be formulated as in (3a)-(3d) but with the objective in (3a) replaced with

$$\text{SW} = \sum_{i=1}^{N} [K_{i,M} u(D(p_{i,M})) + K_{i,S} u(D(p_{i,S}))], \tag{4}$$

assuming the social planner determines the pricing and spectrum allocations for all SPs.

E. Sequential Decision Process

We model the bandwidth and price adjustments of SPs in the network as a two-stage process:

1) Each SP i first determines its bandwidth allocation $B_{i,M}, B_{i,S}$ between macro-cells and small-cells. Denote the aggregate bandwidth allocation profile as $\mathbf{B}$, i.e., $\mathbf{B} = \{B_{i,M}, B_{i,M}, i \in \mathcal{N}\}$.
2) Given $\mathbf{B}$ (assumed known to all SPs), the SPs announce prices for both macro-cells and small-cells. The users then associate with SPs according to the previous user association rule.

This order reflects the fact that bandwidth partitioning takes place over a slower time-scale than price adjustments, since changing the bandwidth partition could conceivably involve reconfiguring equipment at both base stations and handsets, and adjusting the placement of access points along with transmission parameters in order to keep the rate per cell fixed. Adjustment of prices would not require these additional changes.

The two-stage process can be seen as a sequential optimization procedure. That is, the single SP first determines optimal pricing with fixed bandwidth allocation, and then optimizes the bandwidth allocation based on the optimal pricing scheme determined in the first stage.

If there are multiple SPs, the SPs compete for a finite mass of mobile and fixed users, and the two-stage process is thus a sequential game. To gain insight into the resulting allocations we therefore seek a subgame-perfect Nash equilibrium consisting of the following: (*i*) A price equilibrium based on each fixed bandwidth allocation; and (*ii*) A bandwidth allocation equilibrium given that prices are set according to (*i*).[6]

III. Equilibrium Pricing Strategies

In this section we derive the unique price equilibrium for a given bandwidth allocation. Depending on whether macro-cells serve some fixed users or not, two cases are possible:

1. *Mixed service*: Macro-cells serve mobile users and a subset of fixed users;

2. *Separate service*: Macro-cells only serve mobile users.

Theorem 1 (Price Equilibrium): Given a fixed bandwidth allocation profile, $\mathbf{B}$, there exists a unique price equilibrium satisfying the following:

i) The market clears so that the entire mass of users (both mobile and fixed) is served.

[5] See Appendix A for a detailed mathematical description.

[6] For convenience we will often refer to these outcomes in the monopoly setting as an equilibrium too. Additionally, in the competitive case, we will often refer to the overall sub-game perfect Nash equilibrium as simply a Nash equilibrium.

ii) The total rate demand in every macro- or small-cell is equal to the total provisioned capacity.

iii) There exists a threshold $B_{S,0} = \frac{N_f \sum_{i=1}^{N} B_i}{\lambda_S N_m + N_f}$ such that if $B_S = \sum_{i=1}^{N} B_{i,S} < B_{S,0}$, the price equilibrium results in mixed service. Otherwise the separate service case holds.

Theorem 1 indicates that for any given bandwidth allocation, the unique price equilibrium occurs at the *market clearing* price, at which all users are served and all available rate is allocated. As a result, the equilibrium prices can be uniquely determined. For mixed service, the prices are equal across all the cells, whereas for separate service, all macro-cell prices are equal and all small-cells prices are equal, but macro-cell prices are higher than small-cell prices. Note that this result applies to both monopoly and competitive scenarios.

IV. Bandwidth Allocation

We next consider the equilibrium bandwidth allocation between macro-cells and small-cells based on the characterization of the pricing strategy in Theorem 1.

Using Theorem 1, we formulate the bandwidth optimization problem in the separate and mixed service cases, respectively. With separate service, this is given by:

$$\underset{B_{i,M}, B_{i,S}}{\text{maximize}} \quad S_i = B_{i,M} R_0 p_{i,M} + \lambda_S B_{i,S} R_0 p_{i,S}, \tag{5a}$$

$$\text{subject to} \quad D(p_{i,M}) = \frac{\sum_{i=1}^{N} B_{i,M} R_0}{N_m}, \tag{5b}$$

$$D(p_{i,S}) = \frac{\lambda_S \sum_{i=1}^{N} B_{i,S} R_0}{N_f}, \tag{5c}$$

$$\frac{\sum_{i=1}^{N} B_{i,M} R_0}{N_m} \leq \frac{\lambda_S \sum_{i=1}^{N} B_{i,S} R_0}{N_f}, \tag{5d}$$

$$B_{i,M} + B_{i,S} \leq B_i, \tag{5e}$$

$$B_{i,M}, B_{i,S} \geq 0. \tag{5f}$$

The constraints (5b)-(5c) follow from the fact that prices at equilibrium clear the market. Constraint (5d) is the condition that separate service holds. With mixed service, the optimization problem is the same except that the inequality sign in (5d) is reversed.

A. Bandwidth Optimization for Monopoly SP

Comparing the separate and mixed service cases, the following theorem gives the optimal bandwidth allocation for a monopoly SP.

Theorem 2: For a monopoly SP in the revenue optimal bandwidth allocation, all bandwidth is allocated and the separate service scenario occurs.

This theorem shows that a monopolist will always allocate bandwidth so that macro-cells are only used to serve mobile users. Intuitively, since small-cells have a higher spectral efficiency it is advantageous to shift all fixed users to use these.

It is easy to show that at the optimal point, the marginal revenue increase for both services are equal if they are both used. In other words, if B_M and B_S are both greater than zero, then

$$u'(R_M) + R_M u''(R_M) = \lambda_S \big[u'(R_S) + R_S u''(R_S)\big]. \tag{6}$$

Otherwise, $B_S = B$ and $B_M = 0$, where $R_M = \frac{B_M R_0}{N_m}$ and $R_S = \frac{\lambda_S B_S R_0}{N_f}$ are the average rates in macro-cells and small-cells, respectively.

Due to higher spectral efficiency (i.e., $\lambda_S > 1$), when the service rate is equal in macro- and small-cells, the marginal revenue increase due to additional bandwidth is higher in small-cells. As a consequence, at an optimal bandwidth allocation, the fixed users with small-cell service achieve a higher average rate and are subject to a lower price than mobile users with macro-cell service.

B. Bandwidth Equilibrium for Competing SPs

We now study the equilibrium of the bandwidth allocation stage in a competitive scenario with multiple SPs. This will be a bandwidth allocation profile $\mathbf{B}$ such that no SP can increase its revenue by unilaterally changing its own bandwidth allocation, taking into account the price equilibrium and corresponding user association.

Each SP's strategy choice in this game allows it to offer only macro-cell service, only small-cell service or both. Furthermore, the resulting revenue depends on the decisions of the other providers through the resulting prices and user associations. This leads to different scenarios that must be considered.

It is easy to verify that each SP i will always use the total bandwidth B_i. Based on Assumption 1 and using $B_{i,M} = B_i - B_{i,S}$, it can be verified that in both the separate and mixed cases S_i is a concave function of $B_{i,S}$. This enables us to characterize the optimal bandwidth allocation in both cases, which we then use to prove the following theorem.

Theorem 3 (Existence of Nash Equilibrium): A subgame perfect Nash equilibrium always exists for the bandwidth and pricing game and every equilibrium falls into the separate service case.

The proof of this theorem has two steps. We first prove that no Nash equilibrium exists in the mixed service case. We then prove that a Nash equilibrium always exists in the separate service case using Rosen's Theorem [39].

Even under the separate service case, the equilibria can fall into one of the following distinct cases:

1) Small-cell only Nash Equilibrium (SNE): All SPs only allocate bandwidth to small-cells.
2) Macro-Small-cell Nash Equilibrium (MSNE): All SPs allocate bandwidth to both macro- and small-cells.
3) Small-cell Favored Nash Equilibrium (SFNE): A subset of SPs only allocate bandwidth to small-cells and the other SPs allocate bandwidth to both macro- and small-cells.
4) Macro-cell Favored Nash Equilibrium (MFNE): A subset of SPs only allocate bandwidth to macro-cells and the other SPs allocate bandwidth to both macro- and small-cells.

We next show the unique equilibrium must be from the four preceding cases. However, the specific type of the equilibrium depends on specific system parameters.

Theorem 4 (Uniqueness of Nash Equilibrium): The Nash equilibrium of the bandwidth and pricing game is unique and satisfies the conditions in Theorem 3. In equilibrium fixed users (served only by small-cells) achieve a higher average rate than mobile users (served by macro-cells).

Hence, there is no equilibrium in which some set of SPs only allocate bandwidth to small-cells, while some other SPs only allocate bandwidth to macro-cells.

The proof consists of the following four steps. First, we show no Nash equilibrium exists if $R_M = R_S$, where $R_M = \frac{\sum_{i=1}^{N} B_{i,M} R_0}{N_m}$ and $R_S = \frac{\lambda_S \sum_{i=1}^{N} B_{i,S} R_0}{N_f}$ are the average rates in macro-cells and small-cells, respectively. Then by Theorem 3, since the equilibrium falls into the separate service case, it follows that at any Nash equilibrium $R_M < R_S$. Next, we prove that at a Nash equilibrium it never happens that some SPs only allocate bandwidth to small-cells while some SPs only allocate bandwidth to macro-cells. We are therefore left with the preceding four classes of Nash equilibria. In the third step, we prove for a given set of system parameters the equilibrium must fall in only one of the categories in Theorem 3. Finally, we show that the equilibirum must be unique within this class. We point out that we have been unable to find a specific utility function that results in an MFNE,[7] but have also been unable to prove that such an equilibrium does not exist.

C. Equilibria Properties

Next we characterize some properties of the four classes of possible Nash equilibrium, which help us to understand two fundamental questions. The first question is which of them occurs under a given set of parameters, which will be addressed in the remainder of this section. The second question is whether they are socially optimal, which will be addressed in Section V. We also present a numerical example in Section VI to illustrate how the endowments of bandwidth across SPs affects the different types of equilibria that occur.

At an SNE, all SPs only allocate bandwidth to small-cells, which makes this equilibrium easy to characterize. Hence in this section we focus on the three other possibilities: MSNE, MFNE and SFNE.

Proposition 1 (MSNE Properties): The following necessary condition applies at an MSNE:

$$\lambda_S \Big[N u'(R_S) + R_S u''(R_S) \Big] = N u'(R_M) + R_M u''(R_M). \tag{7}$$

Note that when $N = 1$, i.e., the monopoly case, (7) becomes identical to (6).

Further, for any two SPs i and j with total bandwidths B_i and B_j in an MSNE, one of the following two conditions holds:

1) Symmetry: If $B_i = B_j$, then both SPs' bandwidth allocations are the same, i.e., $B_{i,S} = B_{j,S}$, and $B_{i,M} = B_{j,M}$.
2) Monotonicity: If $B_i > B_j$, then SP i allocates more bandwidth to both macro- and small-cells than SP j, i.e., $B_{i,S} > B_{j,S}$, and $B_{i,M} > B_{j,M}$.

Proposition 2 (MFNE and SFNE Properties): The MFNE and SFNE classes have the following properties:

1) At an MFNE, the marginal increase of the sum revenue of all SPs with respect to the small-cell bandwidth is larger than that of macro-cells, while the marginal increase of social welfare with respect to the small-cell bandwidth is smaller than that of macro-cells. At an SFNE, the reverse is true.
2) At an MFNE (SFNE), the SPs that only allocate bandwidth in macro-cells (small-cells) are those that have smaller total bandwidth.

Part 2) of Proposition 2 also yields the following corollary about the equilibrium in the symmetric case.

Corollary 1: In the symmetric case where $B_i = B, \forall i \in \mathcal{N}$, the resulting Nash equilibrium must be either an SNE or an MSNE.

The following Corollary then specifies the transition from SNE to MSNE in the symmetric case.

Corollary 2: In the symmetric case where $B_i = B, \forall i \in \mathcal{N}$, when B is small, the equilibrium is SNE; and when B is large, the equilibrium is MSNE. The transition from SNE to MSNE occurs when B results in rates R_S^0 and R_M^0 that satisfy (7), where $R_S^0 = \frac{NBR_0}{N_f}, R_M^0 = 0$.

According to Proposition 2 and the properties of SNE, we have the following corollary.

Corollary 3: For α-fair utility functions $u(r) = \frac{r^{1-\alpha}}{1-\alpha}$ with $\alpha \in (0, 1)$, only an MSNE exists. For $u(r) = \log(1 + r)$, an MFNE never exists.

Note that different from Corollary 1 and Corollary 2, Corollary 3 does not require any symmetry assumption.

V. Social Welfare

In this section we change the objective to social welfare instead of revenue, and study the corresponding pricing decisions and bandwidth allocations.

For social welfare maximization, the SPs must coordinate resource allocation, hence the corresponding pricing and bandwidth allocation problem is independent of the number of SPs. As a result, the following theorem holds in both the monopoly and competitive scenarios.

Theorem 5: The welfare optimal prices and bandwidth allocation have the following properties:

1) All bandwidth is allocated and prices are set so that the total rate demand is equal to the supply;
2) The bandwidth allocation falls into the separate service case; and
3)
$$u'(R_M) = \lambda_S u'(R_S), \tag{8}$$
when $B_M, B_S > 0$. Otherwise $B_S = B, B_M = 0$.

[7]The difficulty of constructing such a utility function is that the sufficient and necessary conditions for an MFNE to exist are complicated and do not provide a straightforward way to construct a utility.

Note the first two properties are the same as in the individual revenue-maximizing case, while the third property differs in that here the marginal change in welfare with respect to bandwidth is the same for both services, as opposed to the marginal change in revenue. Also, as in the revenue-maximizing case, (8) can be used to show that for the welfare maximizing allocation, the service price in small-cells is lower than the price in macro-cells, and therefore fixed users in small-cells achieve higher average rates than mobile users with macro-cell service.

Applying Corollary 3 and Theorem 5, we find that α-fair utility functions have the following special properties.

Proposition 3: Given α-fair utility functions, in the monopoly scenario the unique optimal bandwidth allocation is always socially optimal. In the competitive scenario, the unique equilibrium, an MSNE, is socially optimal.

We emphasize that Proposition 3 is restricted to α-fair utility functions. For other utility functions, the revenue maximizing bandwidth allocation is not generally socially optimal for any finite number of SPs. However, as the number of SPs increases, competition among SPs may make the revenue and social welfare objectives more closely aligned. Specifically, we consider the asymptotic performance as the number of SPs and users scale linearly with the number of SPs such that the ratio of the total bandwidth to the number of fixed and mobile users stays the same and strictly positive (i.e., the ratio never diminishes to zero). The next theorem characterizes the limit for the different classes of equilibria.

Theorem 6 (Asymptotic Social Welfare Optimality): As the number of SPs $N \to \infty$, the social welfare associated with any SNE or MSNE achieves the maximum value. In contrast, an MFNE or SFNE generally does not achieve the maximum social welfare for any N, or as $N \to \infty$.

The asymptotic social optimality of an SNE and an MSNE is analogous to the asymptotic social optimality for classic Cournot competition [21]. In that scenario, as the number of competing firms increases without bound, with linear demand and constant marginal cost, the Cournot market becomes perfectly competitive. Hence, the price converges to the marginal cost yielding the socially optimal allocation.

The similarity of our model to Cournot competition comes from the fact that SPs compete on quantity (bandwidth in the second stage), and the price is determined by the quantities announced by the firms. However, in the classic Cournot model firms only decide the total amount of a single homogeneous good. In our model, SPs optimize the allocation of a given total amount of bandwidth among two "goods": macro- and small-cell service. We also have two types of users, i.e., mobile users and fixed users. Despite these differences, we see similar asymptotic optimality emerging when $N \to \infty$.

Theorem 6 motivates the question as to when the limiting equilibria is either an MSNE or SNE.

According to Corollary 1, when the SPs have equal bandwidth allocations, the equilibrium becomes socially optimal for large N. A general condition for the limiting equilibria to be an MSNE when SPs have different amounts of bandwidth is not easy to characterize. Instead, to provide insight, we consider a simpler case with two groups of SPs, one with total bandwidth B_1 and the other with total bandwidth B_2. We will assume the specific utility function $u(r) = \log(1+r)$. The number of SPs in the groups are a_1 and a_2, respectively, with $a_1 + a_2 = N$. We then take $N \to \infty$, fixing the fraction of users in each group. That is: $N_f^N = k_1 N$, and $N_m^N = k_2 N$, where k_1, k_2 are positive constants.[8]

Proposition 4: For the preceding scenario, there exists a threshold $B_0 = \frac{(\lambda_S - 1)k_1}{\lambda_S R_0}$, such that for large enough N the equilibrium is an MSNE if and only if $B_1, B_2 \geq B_0$.

Note that the conditions are independent of a_1 and a_2, i.e., independent of how we divide the SPs into the two groups. This result suggests that as long as each SP has a sufficient amount of bandwidth determined by the threshold, it will be able to compete with the other SPs in both macro- and small-cells. When the number of SPs is large, this leads to perfect competition and achieves social optimality.

VI. Case Studies and Numerical Results

In this section we consider some particular examples to illustrate the results obtained in previous sections.

A. Monopoly SP

We first illustrate the bandwidth allocations that maximize revenue and social welfare in the monopoly scenario. For α-fair utility functions, the optimal prices and bandwidth allocations for both revenue and social welfare maximization are the same. Specifically, we have:

$$\frac{R_S^*}{R_M^*} = \lambda_S^{\frac{1}{\alpha}}. \tag{9}$$

As $\alpha \to 0$, $u(r)$ becomes a linear function in which case $R_S^*/R_M^* \to \infty$, and the SP allocates all bandwidth to the small-cells. As $\alpha \to 1$, $u(r)$ becomes $\log(r)$ in which case $R_S^*/R_M^* \to \lambda_S$, and the optimal bandwidth allocation is proportional to the number of users in each cell.

For the $\log(1+r)$ utility function, the bandwidth allocations that maximize revenue and social welfare are different. Specifically,

1) to maximize revenue: $\frac{R_S^*+1}{R_M^*+1} = \sqrt{\lambda_S}$,
2) to maximize social welfare: $\frac{R_S^*+1}{R_M^*+1} = \lambda_S$.

It is easily shown that the R_S^* obtained from maximizing social welfare is greater than the R_S^* obtained from maximizing revenue, whereas the R_M^* obtained from maximizing social welfare is less than the R_M^* that maximizes revenue. As a result, the bandwidth allocation that maximizes social welfare requires the SP to allocate more bandwidth to small-cells.

B. Competing SPs

Next we consider the scenario of multiple competing SPs. Compared with a monopoly SP where the optimal pricing and bandwidth allocation can be computed directly, calculating a Nash equilibrium in the competitive case appears difficult (except in the case of a MSNE, where it can be found by

[8] This is to ensure that no user obtains an infinite service rate.

solving a system of equations), so instead we consider an iterative best-response method.

The best response dynamic we consider is a coordinate gradient method. In particular, each SP changes its strategy in the direction given by the gradient of its revenue function with respect to its bandwidth allocation, subject to proper constraints. If we denote the proportionality constant or step-size for SP i to be $\mu_i > 0$, the update for SP i is:

$$B_{i,S}(n+1) = \left[B_{i,S}(n) + \mu_i \frac{\partial S_i}{\partial B_{i,S}(n)}\right]_{B^0_{i,S}(n)}^{B_i} \tag{10}$$

where $B^0_{i,S}(n)$ is the boundary point at $R_S(n) = R_M(n)$ and $[x]_a^b = \max\big(a, \min(x,b)\big)$.

Theorem 7 (Convergence of Best Response Updates): Starting at any initial point with separate service (i.e., $R_M \leq R_S$), if each SP performs the update in (10) sequentially, the best response updates converge to the unique Nash equilibrium for some appropriate choice of step-sizes $\{\mu_i, i \in \mathcal{N}\}$.

We prove this by applying Rosen's convergence theorem [39, Theorem 10]. That requires showing the symmetric matrix $G + G^T$ is negative definite, where G has elements $G_{ij} = \theta_i \frac{\partial^2 S_i}{\partial B_{i,S} \partial B_{j,S}}$ for some fixed choice of $\theta_i > 0$. [9]

We next use this sequence of best response updates to study Nash equilibria numerically. For all numerical examples, the number of fixed and mobile users scale linearly with the number of SPs, i.e., $N_f^N = k_1 N, N_m^N = k_2 N$.

Figure 1 shows the ratio of the welfare obtained at an MSNE to the socially optimal welfare for two different utility functions[10] versus the number of SPs N. In this scenario we assume all SPs have the same total available bandwidth $B_i = 1$. The system parameters are: $\lambda_S = 4, k_1 = 60, k_2 = 100, R_0 = 50, \theta = 0.5$. Under this set of parameters it is easy to show that the Nash Equilibrium is always an MSNE. We can see that as the number of SPs increases, the social welfare approaches the maximum value.

We next study how the endowments of bandwidth across SPs affects the different types of equilibria that occur. For this we consider the same setting as in Proposition 4, where there are two groups of SPs, where SPs within each group have the same bandwidth (B_1 or B_2). The utility function is $u(r) = \log(1+r)$. For simplicity, we assume each group consists of half of the SPs. The other parameters are: $\lambda_S = 2, k_1 = k_2 = 25, R_0 = 50$. Figures 2 and 3 illustrate the corresponding Nash equilibrium regions as functions of B_1 and B_2, for the cases $N = 2$ and $N = 20$, respectively. We can see that when both B_1 and B_2 are sufficiently small, the Nash equilibrium is an SNE. When one bandwidth is large while the other is small, the Nash equilibrium is an SFNE. When both B_1, B_2 are large enough, the Nash equilibrium is an MSNE. For the given parameters, the threshold in Proposition 4, $B_0 = 0.25$.

[9]From [39] this method also yields an alternate proof for the uniqueness of the Nash equilibrium.

[10]We use the utility function $1 - e^{-\theta r}$ as an illustration even though it violates some properties we used to prove the existence of a sub-game perfect equilibrium. This also shows that these properties are not necessary for an equilibrium to exist.

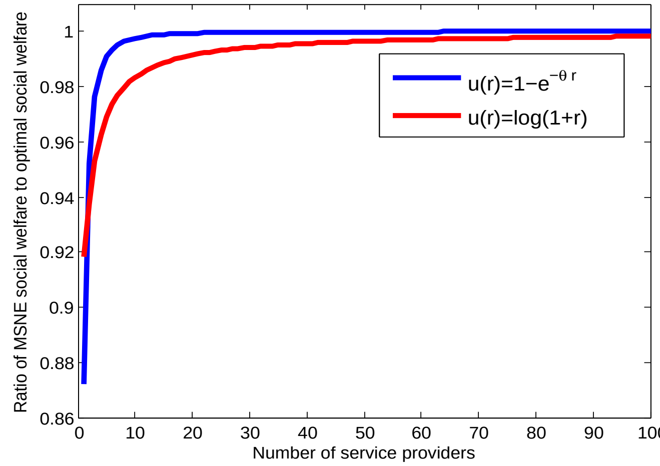

Fig. 1. Ratio of the welfare obtained by an MSNE to the socially optimal welfare versus the number of SPs N.

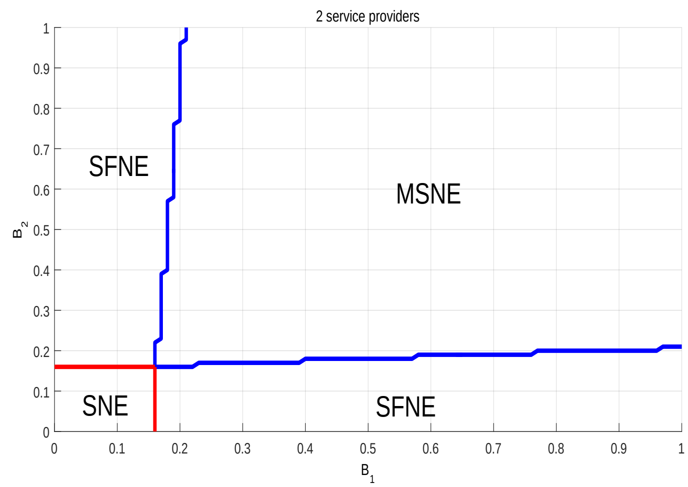

Fig. 2. Nash equilibrium regions as a function of the total bandwidths with two SPs.

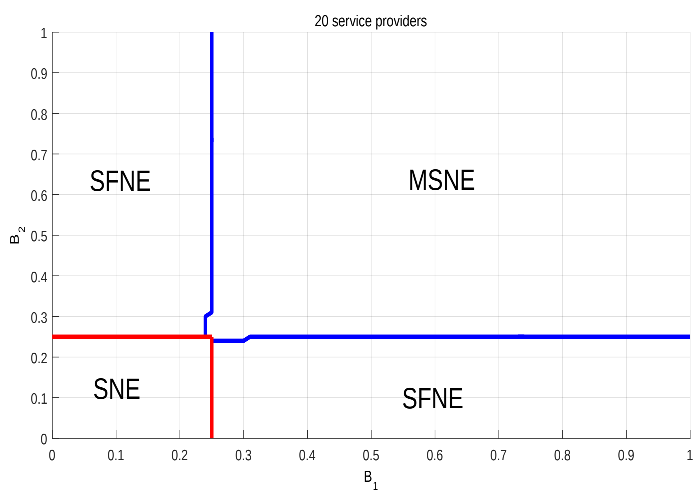

Fig. 3. Nash Equilibrium regions as a function of the total bandwidths with twenty SPs.

Figure 3 shows that with 20 SPs, the boundary is quite close to this asymptotic limit.

These results have the following intuitive explanation. When the available spectrum of both groups is very small, the prices are high in both type of cells. In that case, it is better for SPs to allocate all bandwidth to small-cells since that results in more data rate and therefore more revenue. However, if one group of SPs has a large amount of bandwidth, allocating all bandwidth to small-cells significantly decreases the price in small-cells. Thus, it is beneficial to invest in both small-cells and macro-cells to maximize revenue.

VII. Conclusions

In this paper we investigated service pricing and bandwidth allocation in a heterogeneous wireless network. For both monopoly and competitive scenarios and both revenue and social welfare maximization, we have shown that it is optimal for the macro-cells to serve only mobile users and the small-cells to serve only the fixed users. This conclusion is consistent with the observation that early small-cell deployments have been dominated by indoor systems [19] [20].

In the monopoly scenario, we have characterized the optimal prices for macro- and small-cells along with the optimal bandwidth allocation. It was shown that, in general, revenue and welfare optimization lead to different bandwidth allocations and prices, showing that a revenue maximizing monopolist will cause a welfare loss. The set of α-fair utility functions is an exception in that revenue and social welfare maximization yield the same solution.

We also analyzed the competitive scenario with multiple SPs and showed that a unique Nash equilibrium exists. Again, in general the equilibrium is not socially optimal when each SP maximizes its individual revenue. However, certain classes of Nash equilibria are asymptotically socially optimal when the number of SPs tends to infinity. In order to achieve the benefits of competition, we have to ensure full competition in every active market. Otherwise, even an infinite number of SPs may not yield the socially optimal outcome. Specifically, certain classes of equilibria (MFNE and SFNE) are not asymptotically socially optimal since only a subset of the SPs compete in either the macro- or small-cell markets.

We have made several simplifying assumptions to facilitate our analysis which could be relaxed in future work. For example, we restricted the demand functions of the agents and assumed that macro- and small-cells were priced separately (i.e., SPs did not bundle these). Other properties of spectrum, leading to variable coverage, different spectrum access methods, and heterogeneous traffic requirements might also be analyzed within this framework. Another interesting direction is to account for more dimensions in which users may be heterogeneous, such as mobility patterns and service preferences.

Appendix A
User Association

In this section we formally define how users associate to macro- and small-cells for a given bandwidth allocation and a set of prices chosen by all SPs. We start by sorting the prices of the macro-cells $p_{i,M}$ in ascending order, where ties are broken arbitrarily. Then we associate users via the following steps:

1) Given the priority assigned to mobile users in macro-cells, we start by discussing these users (assuming $N_m > 0$). First consider only the subsequence of macro-cell prices and start with the lowest macro-cell price. All mobile users attempt to attach to the macro-cells of the SP with this price, subject to the provisioned capacity and the available mass of mobile users. If the macro-cells identified belongs to SP i with price $p_{i,M}$, then:
$$K_{i,M} = \min\left(\frac{C_{i,M}}{D(p_{i,M})}, N_m\right) \tag{11}$$
is the mass of mobile users attaching to the macro-cells belonging to SP i. We then recalculate the remaining mass of mobile users by subtracting $K_{i,M}$ from N_m:
$$N_m := N_m - K_{i,M}. \tag{12}$$
We also calculate the residual capacity of the macro-cell:
$$C_{i,M} := C_{i,M} - K_{i,M} D(p_{i,M}). \tag{13}$$
2) If $N_m > 0$, we proceed to the macro-cells with the next lowest price and repeat the preceding procedure.
3) If at any stage multiple SPs' macro-cells have the same price, then we look at them together and allocate the remaining mobile users proportional to the individual rates configured. That is, if $p_{i,M} = p_{j,M} = \cdots = p_{k,M} = p$, for all $i' \in \{i, j, \ldots, k\}$:
$$K_{i',M} = \min\left(\frac{C_{i',M}}{D(p)}, N_m \frac{C_{i',M}}{\sum_{j' \in \{i,j,\ldots,k\}} C_{j',M}}\right). \tag{14}$$
As before we recalculate the remaining mass of mobile users by subtracting $K_{i,M}, K_{j,M}, \ldots, K_{k,M}$ from N_m:
$$N_m := N_m - \sum_{i' \in \{i,j,\ldots,k\}} K_{i',M}. \tag{15}$$
Again we calculate the residual capacities of the macro-cells: for all $i' \in \{i, j, \ldots, k\}$
$$C_{i',M} := C_{i',M} - K_{i',M} D(p). \tag{16}$$
4) The attachment procedure of mobile users stops if either all mobile users are served (after the final attachment $N_m = 0$) or all macro-cells have used up their rate.
5) If all the mobile users get attached, and there exist macro-cells with residual capacity, then we include these with the small-cells and consider the attachment of fixed users. Again we sort the remaining cells, i.e., ones with residual capacity, in increasing order of the access prices.
We start with the cells with the lowest price. If these are small-cells associated with SP i with price $p_{i,S}$, we have:
$$K_{i,S} = \min\left(\frac{C_{i,S}}{D(p_{i,S})}, N_f\right). \tag{17}$$

As before we recalculate the mass of fixed users that are not served yet :

$$N_f = N_f - K_{i,S}. \tag{18}$$

If the cells with the lowest price are macro-cells associated with SP i with price $p_{i,M}$ and residual capacity $C_{i,M}$, we have:

$$K_{i,M} = K_{i,M} + \min\left(\frac{C_{i,M}}{D(p_{i,M})}, N_f\right), \tag{19}$$

where the first term in the min function is the mobile users that are already assigned and the second term is the new fixed users. We also recalculate the mass of fixed users that are not served yet:

$$N_f = N_f - \min\left(\frac{C_{i,M}}{D(p_{i,M})}, N_f\right). \tag{20}$$

6) If $N_f > 0$ and there are cells remaining with positive residual capacity, then we proceed in the same manner as before by picking the cell with the lowest price (with positive residual capacity) and perform the attachment calculations as above. If there are multiple cells (macro-cells or small-cells) with the current lowest price, then we allocate the remaining fixed users proportional to the residual capacities as described in the second step. To be specific, let $\{i,\ldots,j\}$ small-cells and $\{k,\ldots,l\}$ macro-cells have the current lowest price p, then we have: for all $i' \in \{i,\ldots,j\}$

$$\begin{aligned} K_{i',M} = K_{i',M} + \min\Big(&\frac{C_{i',M}}{D(p)}, \\ N_f &\frac{C_{i',M}}{\sum_{j'\in\{i,\ldots,j\}} C_{j',M} + \sum_{l'\in\{k,\ldots,l\}} C_{l',S}}\Big) \end{aligned} \tag{21}$$

and for all $k' \in \{k,\ldots,l\}$

$$\begin{aligned} K_{k',S} = \min\Bigg(&\frac{C_{k',S}}{D(p)}, \\ N_f &\frac{C_{k',S}}{\sum_{j'\in\{i,\ldots,j\}} C_{j',M} + \sum_{l'\in\{k,\ldots,l\}} C_{l',S}}\Bigg). \end{aligned} \tag{22}$$

We also recalculate the mass of fixed users that are not served yet:

$$\begin{aligned} N_f = N_f &- \sum_{l'\in\{k,\ldots,l\}} K_{l',S} \\ &- \sum_{i'\in\{i,\ldots,j\}} \min\Bigg(\frac{C_{i',M}}{D(p)}, \\ &\quad N_f \frac{C_{i',M}}{\sum_{j'\in\{i,\ldots,j\}} C_{j',M} + \sum_{l'\in\{k,\ldots,l\}} C_{l',S}}\Bigg). \end{aligned} \tag{23}$$

7) The attachment procedure stops if either all fixed users are served ($N_f = 0$) or no cells remain with positive residual capacity.

Appendix B
Proof of Theorem 1: Monopoly Case

We first show that the market always clears, i.e., all users are served and the total rate supplied equals the total rate demanded. If there are some mobile users that are not served, the SP can increase the price in macro-cells, then by assumption b) of the concavity of our utility functions, users in macro-cells request less rate, leading to the SP having spare rate to serve more mobile users. The SP can thus use up its rate in macro-cells at a higher price since any unserved mobile users would fill in any spare capacity, which then leads to larger revenue. This argument can be similarly applied to the case where there are some unserved fixed users. The SP can increase the price in small-cells and therefore achieve a higher revenue.

On the other hand, if all users are served but there is still some spare capacity available in macro- or small-cells, the SP can decrease the price in those cells so that users now request a higher rate. By assumption c), $ru'(r)$, which is revenue per user, increases with r, it is easy to see the SP can gain more revenue by doing so.

We now prove that the optimal pricing results in both separate and mixed service scenarios, given a fixed bandwidth allocation. Assume macro-cells only serve mobile users. This then implies that $R_S \geq R_M$; at the boundary $R_S = R_M$, which implies:

$$R_S = \frac{\lambda_S B_S R_0}{N_f} = R_M = \frac{B_M R_0}{N_m}. \tag{24}$$

This can be simplified to:

$$B_S = \frac{N_f B}{\lambda_S N_m + N_f} =: B_S^0. \tag{25}$$

Therefore if B_S is larger than B_S^0, macro-cells only serve mobile users and fixed users only associate with small-cells. In contrast, if B_S is smaller than B_S^0, some fixed users have the incentive to connect to macro-cells. We next prove that is indeed the case at the optimal point.

Suppose $B_S < B_S^0$. If macro-cells only serve mobile users and the market clears, then we have $p_M < p_S$. The SP can then increase the macro-cell price to p'_M, where $p_M < p'_M < p_S$, so that the mobile users obtain a smaller rate, creating some spare capacity. As a result, some fixed users in small-cells would switch to macro-cells. Denote the total mass of customers that switch as δ. The resulting revenue of the SP would then be:

$$S = B_M R_0 u'\Big(R_M \frac{N_m}{N_m+\delta}\Big) + \lambda_S B_S R_0 u'\Big(R_S \frac{N_f}{N_f-\delta}\Big). \tag{26}$$

Differentiating, we then have:

$$\frac{\partial S}{\partial \delta} = -{R'_M}^2 u''(R'_M) + {R'_S}^2 u''(R'_S), \tag{27}$$

where $R'_M = R_M \frac{N_m}{N_m+\delta} = \frac{B_M R_0}{N_m+\delta}$, $R'_S = R_S \frac{N_f}{N_f-\delta} = \frac{\lambda_S B_S R_0}{N_f-\delta}$ are the new per user rates in macro-cells and small-cells, respectively, after the shift of δ mass of fixed users to macro-cells.

Based on our assumptions, $r^2 u''(r)$ decreases with r, therefore as long as $R'_S < R'_M$, i.e., $p'_M < p'_S$, S always increases with δ. As a result, it is always better for macro-cells to serve some fixed users in this case, and the optimal price is $p_M = p_S$.

Appendix C
Proof of Theorem 1: Competitive Case

We first prove that at a price equilibrium, the entire mass of users (both mobile and fixed) will be served. We call the mass of users that are not served free users, and a cell that could serve a free user if capacity was available an eligible cell. As a result, for free mobile users only macro-cells are eligible, while for free fixed users both macro- and small-cells can be eligible cells.

If there are any free users, then we look at an eligible cell with the highest price. The corresponding SP i can increase the price in this cell so that the rate demanded per user decreases. Therefore it has redundant rate available to serve (some of) the free users such that it is provisioned capacity is filled. Thus, it increases its revenue due to the price increase.

We now prove that in every cell the rate demanded should equal the rate supplied. First we establish that at a price equilibrium every cell with leftover capacity will serve some users. Then we show that, in addition, the rate demand will equal the capacity provisioned.

1. We start with the case where there are some small-cells with leftover capacity that have no users attached. Pick one such small-cell of SP i, note that the revenue in this cell is 0. Also, from the assumed properties of the demand the price in this cell must be non-zero. SP i can decrease the price in the small-cell until it equals the price in the next cell that serves some fixed users; we will stop here in most scenarios except for specific cases to be outlined below. If the next cell belongs to another SP j, SP i steals some users from this cell and increases its revenue. The same argument carries through unchanged, if there is more than one such cell. If the next cell is SP i's own macro-cell M_i and not only is its provisioned capacity used up, but it also serves some fixed users, then SP i also changes the price in its macro-cell M_i a little. As a result, a small amount of fixed users in M_i would switch to S_i; it is easy to argue that the macro cell will still exhaust its provisioned capacity. Since $pD(p)$ decreases with p, SP i can increase its revenue. If the macro-cell has some surplus capacity, then depending on whether there are other cells at the same price or not, different strategies can be used to show that the revenue for SP i can be increased. If there are other cells at the same price, then it is sufficient to set the price of the small-cell of SP i equal to the price of the macro-cell of SP i. Then by the user association rule, there will be an increase in the mass of users connecting to SP i, and the increase in revenue follows. If, instead, there are no other cells with the same price, then the SP sets a common price for both its cells with the value slightly below the original price of the macro-cell. Again since $pD(p)$ decreases with p, the revenue increase follows. Note that in this setting SP i is actually indifferent to adding users to its small-cell by the procedure we described or to merely reducing the price in the macro-cell.[11] As the small-cell has non-zero provisioned capacity, we argue that SP i would prefer the former strategy so that the allocated bandwidth does not lie fallow. Similar arguments as above also apply if there are macro-cells M_i that have provisioned capacity but no users attached, but here the price is reduced to the price of the first cell that serves some mobile users.

2. Having established that all cells with non-zero bandwidth will serve some users at a price equilibrium, we proceed to show that configured capacities will also be fully utilized. Consider that there are some small-cells that have some users but also some spare rate. It necessarily follows that all such cells have the same price. Pick one such small-cell of SP i. Then SP i can decrease the price in S_i, and users would request more rate (and also gain more users if there were other cells with the same price) and the spare rate would decrease. Since $pD(p)$ decreases with p, SP i can increase its revenue by doing so. The same logic carries through if there are macro-cells that have some spare rate, irrespective of whether there are only mobile users attached or both mobile and fixed users attached.

Finally, we prove that all macro-cells have the same price and all cells that carry fixed users have the same price. This will then imply our conclusion that fixed users are either served only by small-cells, or if they are also served by macro-cells, then the price is the same across all cells. With equal prices, either across all cells or only for macro- and small-cells separately, our user association rule results in users associating with all cells. Consider sorting the macro-cells in increasing order of their prices. If two adjacent cells (in terms of price) offer different prices, then the SP that owns the cell with the lower price can increase the price in the cell, reduce the rate demand per user in its cell, attract some users from the other cell and thus, increase its revenue (as it will still have no surplus capacity). For the fixed users case, sort all the cells that serve any fixed users in increasing order of their prices. Then the same argument as above shows that this cannot be an equilibrium.

Using similar arguments as stated in Appendix B, we can further establish the price equilibrium in both separate and mixed service scenarios. Combining all these completes the proof.

Appendix D
Proof of Theorem 2

For the mixed service case, the revenue of the SP is:

$$S = (B_M + \lambda_S B_S) R_0 u'\Big[\frac{(B_M + \lambda_S B_S) R_0}{N_m + N_f}\Big]. \tag{28}$$

S is increasing with both B_M and B_F, therefore at the optimal point $B_M + B_S = B$. We then have:

$$S = (N_m + N_f) R u'(R), \tag{29}$$

where $R = \frac{(\lambda_S - 1) B_S + B}{N_f + N_m} R_0$, which is the average rate each mobile user and fixed user achieves.

Based on our assumption that $ru'(r)$ is strictly increasing, it is easy to see that S increases with R. Hence S also increases with B_S and achieves the maximum at the boundary point, $\frac{B_M R_0}{N_m} = \frac{\lambda_S B_S R_0}{N_f}$.

[11] In the next step it would not matter as the price will be decreased in either case until the capacity is used up.

For the separate service case, the revenue of the SP is given by:

$$
\begin{aligned}
S =& B_M R_0 u'(\frac{B_M R_0}{N_m}) + \lambda_S B_S R_0 u'(\frac{\lambda_S B_S R_0}{N_f}) && \text{(30a)}\\
=& N_m R_M u'(R_M) + N_f R_S u'(R_S), && \text{(30b)}
\end{aligned}
$$

where $R_M = B_M R_0/N_m, R_S = \lambda_S B_S R_0/N_f$ are the average service rates in macro-cells and small-cells, respectively. Since $ru'(r)$ is strictly increasing and concave, it is easy to see that at the optimal point $B_S + B_M = B$, and the marginal revenue increase in both services should be equal if they are both used.

Appendix E
Proof of step 1 in Theorem 3

Denote $R = \Big(\sum_{j=1}^{N} B_{j,M} + \lambda_S B_{j,S}\Big) R_0/(N_m + N_f)$ as the average service rate for each mobile and fixed user. Then we have:

$$
\begin{aligned}
S_i =& (B_{i,M} + \lambda_S B_{i,S}) R_0 u'(R)\\
=& (N_m + N_f) R u'(R) - \Big(\sum_{j\neq i}^{N} B_{j,M} + \lambda_S B_{j,S}\Big) R_0 u'(R).
\end{aligned} \tag{31}
$$

For any fixed $B_{j,M}, B_{j,S}, j \neq i$, $Ru'(R)$ is increasing with R and $u'(R)$ is decreasing with R. Therefore S_i increases with R. However, R also increases with $B_{i,S}$ since $\lambda_S > 1$. Therefore S_i increases with $B_{i,S}$, which indicates in the mixed service scenario all SPs have the incentive to increase the bandwidth allocation to small-cells to increase their revenue, until it reaches the boundary point $R_M = R_S$. However, in Appendix G we will prove that it is impossible that a Nash Equilibrium exists on the boundary point at $R_S = R_M$. Thus, it is not possible for a Nash Equilibrium to exist in the mixed service case.

Appendix F
Proof of step 2 in Theorem 3

In the original game, for any SP i, the bandwidth allocation strategy profile $B_{i,M}, B_{i,S}$ should be in the range of $[0, B_i]$, independent of all other SPs' bandwidth allocation. However, since we have proved no Nash equilibrium exists in the mixed service case, we can transform the original game to a new game in which only separate service holds. In this setting, every SP i shares the same coupled constraint $R_S \geq R_M$. Therefore each SP's strategy profile also depends on the other SPs' bandwidth allocation. Specifically, given all other SPs' bandwidth allocation profile $B_{j,M}, B_{j,S}, j \neq i$, the bandwidth allocation strategy profile for SP i in this generalized game should be:

$$
B_{i,S} \in [\max(0, B_{i,S}^0), B_i], \quad B_{i,M} = B_i - B_{i,S} \tag{32}
$$

where $B_{i,S}^0 = \frac{N_f B_i + N_f \sum_{j\neq i}^{N} B_{j,M} - \lambda_S N_m \sum_{j\neq i}^{N} B_{j,S}}{\lambda_S N_m + N_f}$. That is, when $B_{i,S} = B_{i,S}^0$, $R_S = R_M$.

We have showed that for any SP i, its revenue S_i is a concave function in $B_{i,S}$ in region B given any fixed $B_{-i,S}$. The constraint set is also convex in $\mathbf{B}$. Therefore applying Rosen's Theorem on concave games [39], there always exists a Nash equilibrium in our generalized game setting.

Appendix G
Proof of step 1 in Theorem 4

We first define some notation that divides the SPs at NE into three groups, according to their bandwidth allocation decisions:

$$
\begin{aligned}
G_S =& \{k \in \mathcal{N} | B_{i,S} = B_i\}, && \text{(33a)}\\
G_{MS} =& \{k \in \mathcal{N} | 0 < B_{i,S}, B_{i,M} < B_i\}, && \text{(33b)}\\
G_M =& \{k \in \mathcal{N} | B_{i,M} = B_i\}. && \text{(33c)}
\end{aligned}
$$

That is, G_S (G_M) is the set of SPs that only allocate bandwidth to small-cells (macro-cells). G_{MS} is the set of SPs that allocate bandwidth to both small-cells and macro-cells. Obviously G_S, G_M and G_{MS} are disjoint and $G_S \cup G_M \cup G_{MS} = \mathcal{N}$.

We claim that for SPs in the three sets at Nash Equilibrium, we have:

$$
\begin{aligned}
i \in G_S \Rightarrow D_i \geq 0, && \text{(34a)}\\
i \in G_{MS} \Rightarrow D_i = 0, && \text{(34b)}\\
i \in G_M \Rightarrow D_i \leq 0, && \text{(34c)}
\end{aligned}
$$

where

$$
\begin{aligned}
D_i = \frac{\partial S_i}{\partial B_{i,S}} =& R_0 \Big[\lambda_S u'(R_S) + \frac{\lambda_S^2 B_{i,S} R_0 u''(R_S)}{N_f}\\
& - u'(R_M) - \frac{B_{i,M} R_0 u''(R_M)}{N_m}\Big].
\end{aligned} \tag{35}
$$

The claim follows as D_i decreases with $B_{i,S}$.

Suppose there exists a Nash Equilibrium at $R_S = R_M$. Then we have:

(1) For SP $i \in G_S$,

$$
B_{i,S} = B_i, \forall i \in G_S. \tag{36}
$$

(2) For SP $j \in G_M \cup G_{MS}$, we have:

$$
\begin{aligned}
D_j \leq 0 \Rightarrow& \lambda_S u'(R_S) + \frac{\lambda_S^2 B_{j,S} R_0 u''(R_S)}{N_f}\\
& - u'(R_M) - \frac{B_{j,M} R_0 u''(R_M)}{N_m} \leq 0.
\end{aligned} \tag{37}
$$

Rearranging the items using the definition of R_S and R_M, we have:

$$
\begin{aligned}
=& \Big(\lambda_S u'(R_S) + \lambda_S R_S u''(R_S) -\\
& u'(R_M) - R_M u''(R_M) -\\
& \lambda_S^2 \sum_{k\neq j}^{N} \frac{B_{k,S} R_0}{N_f} u''(R_S) + \sum_{k\neq j}^{N} \frac{B_{k,M} R_0}{N_m} u''(R_M)\Big)\\
\leq& 0.
\end{aligned} \tag{38}
$$

Since $R_S = R_M$, $u''(R_S) = u''(R_M) < 0$, $u'(R_S) + R_S u''(R_S) > 0$, $\lambda_S > 1$, we have:

$$\begin{aligned}
&\lambda_S^2 \sum_{k\neq j}^{N} \frac{B_{k,S}R_0}{N_f} u''(R_S) - \sum_{k\neq j}^{N} \frac{B_{k,M}R_0}{N_m} u''(R_M) \\
&\geq (\lambda_S - 1)\Big(u'(R_S) + R_S u''(R_S)\Big) > 0, \qquad (39)\\
\Rightarrow &\lambda_S^2 \sum_{k\neq j}^{N} \frac{B_{k,S}R_0}{N_f} < \sum_{k\neq j}^{N} \frac{B_{k,M}R_0}{N_m}, \qquad (40)\\
\Rightarrow &\lambda_S \sum_{k\neq j}^{N} \frac{B_{k,S}R_0}{N_f} < \sum_{k\neq j}^{N} \frac{B_{k,M}R_0}{N_m}. \qquad (41)
\end{aligned}$$

However, since $R_S = R_M$, we have:

$$\lambda_S \sum_{k\neq j}^{N} \frac{B_{k,S}R_0}{N_f} + \lambda_S \frac{B_{j,S}R_0}{N_f} = \sum_{k\neq j}^{N} \frac{B_{k,M}R_0}{N_m} + \frac{B_{j,M}R_0}{N_m}. \qquad (42)$$

Therefore, we have:

$$\lambda_S \frac{B_{j,S}R_0}{N_f} > \frac{B_{j,M}R_0}{N_m}, \forall j \in G_M \cup G_{MS}. \qquad (43)$$

However, this means:

$$\begin{aligned}
R_S = &\sum_{i\in G_S} \frac{\lambda_S B_{i,S}R_0}{N_f} + \sum_{j\in G_M\cup G_{MS}} \frac{\lambda_S B_{j,S}R_0}{N_f} \\
>R_M = &\sum_{i\in G_S} \frac{B_{i,M}R_0}{N_m} + \sum_{j\in G_M\cup G_{MS}} \frac{B_{j,M}R_0}{N_m}, \qquad (44)
\end{aligned}$$

which contradicts the assumption that $R_S = R_M$.

Therefore it is impossible that a Nash Equilibrium exists on the boundary point at $R_S = R_M$.

Appendix H
Proof of step 2 in Theorem 4

Suppose there exists one SP i that only allocates bandwidth to macro-cells, and another SP j that only allocates bandwidth to small-cells, then we have:

$$\begin{aligned}
&B_{i,M} = B_i, B_{i,S} = 0, B_{j,M} = 0, B_{j,S} = B_j, \qquad (45a)\\
&D_i \leq 0, D_j \geq 0. \qquad (45b)
\end{aligned}$$

It is easy to see:

$$\lambda_S^2 \frac{B_{j,S}R_0}{N_f} u''(R_S) \geq -\frac{B_{i,M}R_0}{N_m} u''(R_M). \qquad (46)$$

Since $u''(r) < 0$, (46) cannot hold, and we have an obvious contradiction.

Appendix I
Proof of step 3 in Theorem 4

We prove the four classes of Nash Equilibrium are mutually exclusive in the following steps. In the proof we will use some properties of the Nash Equilirbium described in Section IV. These properties are proved in later parts of the Appendix.

A. An MFNE cannot coexist with an SNE, SFNE or MSNE

Suppose for a given set of parameters, there exists one MFNE with bandwidth allocation profile $\mathbf{B}$ and another NE which belongs to SNE, SFNE or MSNE with bandwidth allocation profile $\mathbf{B}'$. Then we have:

$$N\lambda_S u'(R_S) + \lambda_S R_S u''(R_S) \leq N u'(R_M) + R_M u''(R_M), \qquad (47a)$$

$$N\lambda_S u'(R'_S) + \lambda_S R'_S u''(R'_S) \geq N u'(R'_M) + R'_M u''(R'_M). \qquad (47b)$$

Since $u'(r)$ and $u'(r) + r u''(r)$ are both decreasing with r, and $(R_S, R_M), (R'_S, R'_M)$ correspond to two different bandwidth allocation profiles with the same amount of total bandwidth, we can conclude that:

$$R'_S \leq R_S, R'_M \geq R_M. \qquad (48)$$

Then there must exist some SPs $j \in G_{MS}$ in the MFNE that decrease their bandwidth allocation in small-cells $B_{j,S}$ so that it is possible to make $R'_S \leq R_S$. Denote this user group as G_{S-}. For SPs j in group G_{S-}, since they decrease their $B_{j,S}$, at the new NE, they can only be in the user group G'_{MS} or G'_M. Then we have:

$$\begin{aligned}
&G_{S-} = \{j \in G_{MS} | B'_{j,S} < B_{j,S}\}, \qquad (49a)\\
&\forall j \in G_{S-}, j \in G'_{MS} \text{ or } j \in G'_M. \qquad (49b)
\end{aligned}$$

By the definitions for user groups, we have:

$$\begin{aligned}
&\forall j \in G_{S-}, D_j = 0, D'_j \leq 0 \qquad (50a)\\
\Rightarrow &\sum_{j\in G_{S-}} D'_j \leq \sum_{j\in G_{S-}} D_j = 0. \qquad (50b)
\end{aligned}$$

Letting $|G_{S-}| = L$, we have:

$$\begin{aligned}
&L\lambda_S u'(R_S) + \lambda_S^2 \sum_{j\in G_{S-}} \frac{B_{j,S}R_0}{N_f} u''(R_S) = \\
&L u'(R_M) + \sum_{j\in G_{S-}} \frac{B_{j,M}R_0}{N_m} u''(R_M). \qquad (51)
\end{aligned}$$

Similarly, we also have:

$$\begin{aligned}
&L\lambda_S u'(R'_S) + \lambda_S^2 \sum_{j\in G_{S-}} \frac{B_{j,S}R_0}{N_f} u''(R'_S) \leq \\
&L u'(R'_M) + \sum_{j\in G_{S-}} \frac{B_{j,M}R_0}{N_m} u''(R'_M). \qquad (52)
\end{aligned}$$

Rearranging some of the terms, we have:

$$\begin{aligned}
&L\lambda_S u'(R_S) + \lambda_S R_S u''(R_S) = L u'(R_M) + R_M u''(R_M) + \\
&\lambda_S^2 \sum_{k\notin G_{S-}} \frac{B_{k,S}R_0}{N_f} u''(R_S) - \sum_{k\notin G_{S-}} \frac{B_{k,M}R_0}{N_m} u''(R_M), \quad (53)
\end{aligned}$$

and

$$\begin{aligned}
&L\lambda_S u'(R'_S) + \lambda_S R'_S u''(R'_S) \leq L u'(R'_M) + R'_M u''(R'_M) + \\
&\lambda_S^2 \sum_{k\notin G_{S-}} \frac{B_{k,S}R_0}{N_f} u''(R'_S) - \sum_{k\notin G_{S-}} \frac{B_{k,M}R_0}{N_m} u''(R'_M). \quad (54)
\end{aligned}$$

However, note that $R_S' \leq R_S, R_M' \geq R_M$, $u'(r)$ and $u'(r)+xu''(r)$ are both decreasing with r. Since $u''(r)$ is negative and increases with r, for SPs that are not in the user group G_{S-}, we have:

$$\forall k \notin G_{S-}, B_{k,S}' \geq B_{k,S}, B_{k,M}' \leq B_{k,S}. \tag{55}$$

Combining these facts and equation (53) we can conclude:

$$L\lambda_S u'(R_S') + \lambda_S R_S' u''(R_S') > Lu'(R_M') + R_M' u''(R_M') + \lambda_S^2 \sum_{k \notin G_{S-}} \frac{B_{k,S} R_0}{N_f} u''(R_S') - \sum_{k \notin G_{S-}} \frac{B_{k,M} R_0}{N_m} u''(R_M'). \tag{56}$$

The strict inequality comes from the following argument. If $R_S' < R_S$, the strict inequality is trival. If $R_S' = R_S$, it implies that at least one of the inequalities in (55) must be strict (otherwise $R_S' = R_S$ cannot hold because we have at least one SP in G_{S-}). Thus, we always have a strict inequality in (56). However, (56) contradicts (54). Therefore, an MFNE cannot coexist with an SNE, SFNE or MSNE.

B. An SFNE cannot coexist with an MSNE or SNE

The proof that an SFNE cannot coexist with an MSNE follows using the same argument as the proof in last subsection where we prove that an MFNE cannot coexist with an SNE, SFNE or MSNE.

The proof that an SFNE cannot coexist with an SNE is done similarly. If there also exists an NE belonging to SNE with bandwidth allocation profile $\mathbf{B}$, then we need to have:

$$R_S' \geq R_S, R_M' \leq R_M. \tag{57}$$

Now instead of focusing on the group in which SPs decrease $B_{j,S}$, we need to consider the group in which SPs increase $B_{j,S}$. That is:

$$G_{S+} = \{j \in G_{MS} | B_{j,S}' > B_{j,S}\}. \tag{58}$$

Then we apply the same procedure as before and obtain similar contradictions.

C. An MSNE cannot coexist with an SNE

If there exists an MSNE with bandwidth allocation profile $\mathbf{B}$ and another SNE with bandwidth allocation profile $\mathbf{B}'$, then we must have:

$$R_S' > R_S, R_M' < R_M, \tag{59a}$$

$$N\lambda_S u'(R_S) + \lambda_S R_S u''(R_S) = Nu'(R_M) + R_M u''(R_M), \tag{59b}$$

$$N\lambda_S u'(R_S') + \lambda_S R_S' u''(R_S') > Nu'(R_M') + R_M' u''(R_M'). \tag{59c}$$

However, since $u'(r)$ and $u'(r) + ru''(r)$ are both decreasing with r, by (59a) and (59b) we can conclude:

$$N\lambda_S u'(R_S') + \lambda_S R_S' u''(R_S') < Nu'(R_M') + R_M' u''(R_M'). \tag{60}$$

Clearly we have a contradiction.

APPENDIX J
PROOF OF STEP 4 IN THEOREM 4

We prove the uniqueness of the Nash Equilibrium in each of the four classes one by one.

A. Uniqueness of an SNE

The uniqueness of an SNE is straightforward since for an SNE, we have:

$$\forall i \in \mathcal{N}, B_{i,M} = 0, B_{i,S} = B_i, D_i \geq 0. \tag{61}$$

B. Uniqueness of an MFNE

The proof of uniqueness of an MFNE is very similar to the proof of the property that an MFNE cannot coexist with an SNE, SFNE or MSNE. Suppose there exist two MFNEs with bandwidth allocation profiles $\mathbf{B}$ and $\mathbf{B}'$, respectively. Basically we need to consider two scenarios:

- $R_S' \leq R_S$
  In this case we simply use the same arguments as in Appendix I to get the contradiction.
- $R_S' > R_S$
  In this case proof is similar to the preceding case. The difference is that we now focus on the group of SPs that increase $B_{j,S}$.

$$G_{S+} = \{j \in G_{MS} | B_{j,S}' > B_{j,S}\}, \tag{62a}$$

$$\forall j \in G_{S+}, j \in G, M \text{ or } G_{MS}, j \in G_{MS}', \tag{62b}$$

$$\sum_{j \in G_{S+}} D_j \leq \sum_{j \in G_{S+}} D_j' = 0. \tag{62c}$$

Then applying the same procedure as in Appendix I, denoting $|G_{S+}| = L$, we need to have:

$$L\lambda_S u'(R_S) + \lambda_S R_S u''(R_S) \leq Lu'(R_M) + R_M u''(R_M) + \lambda_S^2 \sum_{k \notin G_{S+}} \frac{B_{k,S} R_0}{N_f} u''(R_S) - \sum_{k \notin G_{S+}} \frac{B_{k,M} R_0}{N_m} u''(R_M), \tag{63a}$$

$$L\lambda_S u'(R_S') + \lambda_S R_S' u''(R_S') = Lu'(R_M') + R_M' u''(R_M') + \lambda_S^2 \sum_{k \notin G_{S+}} \frac{B_{k,S}' R_0}{N_f} u''(R_S') - \sum_{k \notin G_{S+}} \frac{B_{k,M}' R_0}{N_m} u''(R_M'). \tag{63b}$$

However, we also have:

$$\forall k \notin G_{S+}, B_{k,S}' \leq B_{k,S}, B_{k,M}' \geq B_{k,S}. \tag{64}$$

Combining with $R_S' > R_S, R_M' < R_M$ and the facts that $u'(r)$ and $u'(r) + ru''(r)$ are both decreasing with r and that $u''(r)$ is negative and increases with r, we can derive:

$$L\lambda_S u'(R_S') + \lambda_S R_S' u''(R_S') < Lu'(R_M') + R_M' u''(R_M') + \lambda_S^2 \sum_{k \notin G_{S-}} \frac{B_{k,S}' R_0}{N_f} u''(R_S') - \sum_{k \notin G_{S-}} \frac{B_{k,M}' R_0}{N_m} u''(R_M'). \tag{65}$$

Clearly we have a contradiction.

C. Uniqueness of an SFNE

The proof of uniqueness of an SFNE is essentially the same as that for an MFNE. We can apply similar procedures as in the preceding section and get a contradiction if there exist two SFNEs with different bandwidth allocation profiles.

D. Uniqueness of an MSNE

We can still apply the arguments used to prove the uniqueness of an MFNE and SFNE to prove the uniqueness of an MSNE. However, we provide another method here. This alternative method reveals many significant properties of an MSNE.

We know that for an MSNE, we need to have:

$$\forall i \in \mathcal{N}, B_{i,M} > 0, B_{i,S} > 0, D_i = 0 \Rightarrow \sum_{i=1}^{N} D_i = 0. \quad (66)$$

Based on the expression for D_i, we have:

$$N\lambda_S u'(R_S) + \lambda_S R_S u''(R_S) = Nu'(R_S) + R_M u''(R_M). \quad (67)$$

Next, find any distinct pair of SPs, and calculate $D_i - D_j$. Then we get:

$$\lambda_S^2 \frac{(B_{i,S} - B_{j,S})R_0}{N_f} u''(R_S) = \frac{(B_{i,M} - B_{j,M})R_0}{N_m} u''(R_M). \quad (68)$$

Thus, we can conclude that the MSNE must satisfy the following system of equations:

$$\begin{cases} \lambda_S[Nu'(R_S) + R_S u''(R_S)] = Nu'(R_M) + R_M u''(R_M), \\ \frac{\lambda_S^2 \Delta B_{ij,S}}{N_f} u''(R_S) = \frac{\Delta B_{ij,M}}{N_m} u''(R_M), \forall i,j \in \mathcal{N}, i \neq j, \end{cases} \quad \text{(P1)}$$

where $\Delta B_{ij,S} = B_{i,S} - B_{j,S}$, $\Delta B_{ij,M} = B_{i,M} - B_{j,M}$.

By the monotonicity of both $u'(r)$ and $u'(r) + ru''(r)$, we conclude that $Nu'(r) + ru''(r)$ also monotonically decreases with r. Then by the first equation we can uniquely determine $\sum_{i=1}^{N} B_{i,S}$. The second equation characterizes the relationships of $B_{i,S}$ between any pair of SPs. The preceding system of equations has N unknowns and N independent linear equations. Thus if there is a solution, it must be unique.

Appendix K
Proof of Proposition 1

At an MSNE, all SPs allocate bandwidth to both macro-cells and small-cells. This allocation is the solution of the following system of equations:

$$\begin{cases} \lambda_S[Nu'(R_S) + R_S u''(R_S)] = Nu'(R_M) + R_M u''(R_M), \\ \frac{\lambda_S^2 \Delta B_{ij,S}}{N_f} u''(R_S) = \frac{\Delta B_{ij,M}}{N_m} u''(R_M), \forall i,j \in \mathcal{N}, i \neq j, \end{cases} \quad \text{(P1)}$$

where $\Delta B_{ij,S} = B_{i,S} - B_{j,S}$ is the difference in bandwidth allocation to small-cells between SP i and SP j, and the analogous definition applies to $\Delta B_{ij,M}$. We next use these equations to prove the two properties in the theorem:

1) When $B_i = B_j, i,j \in \{1,2,...,N\}$, then $\Delta B_{ij,S} = -\Delta B_{ij,M}$. According to the second equation, $\Delta B_{ij,S} = \Delta B_{ij,M} = 0$.
2) When $B_i > B_j, i,j \in \{1,2,...,N\}$, then according to the second equation, $\Delta B_{ij,S}, \Delta B_{ij,M} > 0$.

Appendix L
Proof of Proposition 2

1) For an MFNE, we have:

$$\forall i \in G_M, \forall j \in G_{MS} \Rightarrow D_i \leq D_j = 0. \quad (69)$$

Therefore based on the defition of D_i (35), we have:

$$\lambda_S u'(R_S) - u'(R_M) \leq \frac{B_{i,M} R_0 u''(R_M)}{N_m} - \frac{\lambda_S^2 B_{i,S} R_0 u''(R_S)}{N_f} \quad \text{(70a)}$$

$$= \frac{B_{i,M} R_0 u''(R_M)}{N_m} < 0. \quad \text{(70b)}$$

Then we have:

$$\lambda_S u'(R_S) < u'(R_M). \quad (71)$$

However, we also have:

$$\sum_{j \in G_{MS}} D_j = 0, |G_{MS}| = L \Rightarrow$$

$$L\lambda_S u'(R_S) + \lambda_S R_S u''(R_S) - Lu'(R_M) - R_M u''(R_M)$$
$$= \lambda_S^2 \sum_{k \in G_M} \frac{B_{k,S} R_0}{N_f} u''(R_S) - \sum_{k \in G_M} \frac{B_{k,M} R_0}{N_m} u''(R_M) \quad \text{(72a)}$$

$$= -\sum_{k \in G_M} \frac{B_{k,M} R_0}{N_m} u''(R_M) > 0. \quad \text{(72b)}$$

Finally, we have:

$$L\lambda_S u'(R_S) + \lambda_S R_S u''(R_S) > Lu'(R_M) + R_M u''(R_M). \quad (73)$$

Since we have $\lambda_S u'(R_S) < u'(R_M)$, we obtain the MFNE condition:

$$\lambda_S[u'(R_S) + R_S u''(R_S)] > u'(R_M) + R_M u''(R_M), \quad \text{(74a)}$$
$$\lambda_S u'(R_S) < u'(R_M). \quad \text{(74b)}$$

For an SFNE, we apply the same procedures and get the condition:

$$\lambda_S u'(R_S) + \lambda_S R_S u''(R_S) < u'(R_M) + R_M u''(R_M), \quad \text{(75a)}$$

$$\lambda_S u'(R_S) > u'(R_M). \quad \text{(75b)}$$

2) For an MFNE, utilizing the fact that $D_i \leq D_j = 0$, we have:

$$\lambda^2 \sum_{k \neq i}^{N} \frac{B_{k,S} R_0}{N_f} u''(R_S) - \sum_{k \neq i}^{N} \frac{B_{k,M} R_0}{N_m} u''(R_M) \geq \lambda^2 \sum_{k \neq j}^{N} \frac{B_{k,S} R_0}{N_f} u''(R_S) - \sum_{k \neq j}^{N} \frac{B_{k,M} R_0}{N_m} u''(R_M) \quad \text{(76a)}$$

$$\Rightarrow \lambda^2 \frac{B_{j,S} R_0}{N_f} u''(R_S) - \frac{B_{j,M} R_0}{N_m} u''(R_M) \geq \lambda^2 \frac{B_{i,S} R_0}{N_f} u''(R_S) - \frac{B_{i,M} R_0}{N_m} u''(R_M) \quad \text{(76b)}$$

$$\Rightarrow \lambda^2 \frac{B_{j,S} R_0}{N_f} u''(R_S) - \frac{B_{j,M} R_0}{N_m} u''(R_M) \geq -\frac{B_i R_0}{N_m} u''(R_M). \quad \text{(76c)}$$

Since $u''(r) < 0$, it is clear that we need to have:

$$B_{j,M} > B_i. \tag{77}$$

For an SFNE, we apply the same procedures, and get the analogous results.

Appendix M
Proof of Corollary 3

For α-fair utility functions, $ru''(r) + u'(r) = (1-\alpha)r^{-\alpha} > 0$ and $u'(r) = r^{-\alpha}$. Therefore neither (74a)-(74b) nor (75a)-(75b) can occur. On the other hand, the marginal revenue increase in macro-cells, $R_M u''(R_M) + u'(R_M)$, goes to infinity when R_M goes to 0, which means an SFNE never exists.

Appendix N
Proof of Theorem 5

The proof of the welfare optimal pricing is very similar to the proof we presented in Appendix B. We do not repeat the steps here.

As for the bandwidth allocation, for the mixed service case, we have:

$$\text{SW} = (N_m + N_f)u(R), \tag{78}$$

where $R = \frac{(B_M + \lambda_S B_S)R_0}{N_m + N_f}$ is the average service rate in both macro-cells and small-cells. It is easy to see $B_M + B_S = B$ and social welfare increases with R. Since $\lambda_S > 1$, social welfare increases with B_S and the maximum occurs at the boundary point, $B_M R_0/N_m = \lambda_S B_S R_0/N_f$.

For the separate service case, we have:

$$\text{SW} = N_m u(R_M) + N_f u(R_S), \tag{79}$$

where $R_M = B_M R_0/N_m, R_S = \lambda_S B_S R_0/N_f$ are the average service rates in macro-cells and small-cells, respectively. Since $u(r)$ in increasing and strictly concave, the statements in Theorem 5 are easy to verify.

Appendix O
Proof of Theorem 6

We first prove the following lemma.

Lemma 1 (Characterization of Limiting NE): Denote $R_S = \lambda_S \lim_{N\to\infty} \sum_{i=1}^{N} B_{i,S}/N_f$, $R_M = \lim_{N\to\infty} \sum_{i=1}^{N} B_{i,M}/N_m$. Since in our setting we let the mass of users scale linearly with the number of SPs and the ratio of the total bandwidth to the mass of mobile and fixed users stay the same, the limits always exist.

1) At the limiting SNE, we have the following inequality:

$$\lambda_S u'(R_S) \geq u'(R_M). \tag{80}$$

2) At the limiting MSNE, we have:

$$\lambda_S u'(R_S) = u'(R_M). \tag{81}$$

3) We also have:

$$\text{Limiting MFNE}: \lambda_S u'(R_S) \leq u'(R_M), \tag{82}$$
$$\text{Limiting SFNE}: \lambda_S u'(R_M) \geq u'(R_M). \tag{83}$$

Proof: When $N \to \infty$, considering the first equation in the system of equations (P1), we have:

$$\lambda_S u'(R_S) + \frac{\lambda_S R_S u''(R_S)}{N} = u'(R_M) + \frac{R_M u''(R_M)}{N}. \tag{84}$$

Under our assumptions, $u'(r) < \infty, \forall x > 0$. $|ru''(r)| < u'(r), \forall r \geq 0$. Since $R_S > 0$ always holds, as long as $R_M > 0$, $R_S u''(R_S)$ and $R_M u''(R_M)$ are always finite. The equation is thus simplified to be:

$$\lambda_S u'(R_S) = u'(R_M).$$

When $R_M = 0$, if $u'(0)$ is finite, then $R_M u''(R_M)$ is also finite and we are done. When $u'(0)$ is infinite, however, there will not be any solution to the equation system (P1) since $\lambda_S u'(R_S) < \infty$, which then completes the proof. Applying the preceding procedures, we can obtain the properties of the limiting SNE and the limiting MFNE or SFNE. ■

The condition for maximizing social welfare is:

$$\lambda_S u'(R_S) = u'(R_M), \tag{85}$$

if there exists such a solution. If the solution does not exist, the condition is simply $B_{i,S} = B_i, \forall i \in \mathcal{N}$. Clearly the limiting SNE and MSNE satisfy the condition and therefore they are social welfare-optimal. In contrast, generally the limiting MFNE or SFNE do not satisfy the condition.

Appendix P
Proof of Proposition 4

For utility function $u(r) = \log(1 + r)$, $u'(x) = \frac{1}{r+1}, u''(x) = -\frac{1}{(r+1)^2}$. In order to get a limiting MSNE, we need to make sure the following system of equations has a feasible solution:

$$\begin{cases} \lambda_S u'(R_S) = u'(R_M) \\ \frac{\lambda_S^2 \Delta B_{ij,S}}{N_f} u''(R_S) = \frac{\Delta B_{ij,M}}{N_m} u''(R_M), \forall i,j \in \mathcal{N}, i \neq j. \end{cases} \tag{P2}$$

We also have the following:

$$N_f = k_1 N, N_m = k_2 N, a_1 + a_2 = N, \tag{86a}$$
$$R_S = \frac{\lambda_S(a_1 B_{1,S} + a_2 B_{2,S})R_0}{N_f}, \tag{86b}$$
$$R_M = \frac{(a_1 B_{1,M} + a_2 B_{2,M})R_0}{N_m}, \tag{86c}$$
$$B_{1,M} = B_1 - B_{1,S}, B_{2,M} = B_2 - B_{2,S}. \tag{86d}$$

The systems in (P2) can then be simplified to:

$$\begin{cases} a_1 B_{1,S} + a_2 B_{2,S} = \frac{(\lambda_S - 1)N_m N_f + \lambda_S N_f R_0(a_1 B_1 + a_2 B_2)}{\lambda_S(N_m + N_f)R_0} \\ B_{1,S} - B_{2,S} = \frac{N_f(B_1 - B_2)}{N_m + N_f}. \end{cases}$$

It follows that the solutions are given by

$$\begin{cases} B_{1,S} = \frac{(\lambda_S - 1)N_m N_f + \lambda_S N_f B_1 R_0(a_1 + a_2)}{\lambda_S(N_m + N_f)(a_1 + a_2)R_0} \\ B_{2,S} = \frac{(\lambda_S - 1)N_m N_f + \lambda_S N_f B_2 R_0(a_1 + a_2)}{\lambda_S(N_m + N_f)(a_1 + a_2)R_0}. \end{cases}$$

Next we check the feasibility of the solutions. First of all, for the first equation in (P2) to hold, we need to make sure that:

$$\lambda_S u'(R_S) \leq u'(R_M), \text{ when } B_{1,S} = B_1, B_{2,S} = B_2 \tag{87a}$$
$$\Rightarrow a_1 B_1 + a_2 B_2 \geq \frac{(\lambda_S - 1)N_f}{\lambda_S R_0}. \tag{87b}$$

Since $\lambda_S > 1$, this condition also guarantees that $R_S > R_M$.

On the other hand, $B_{1,S}, B_{2,S}$ should lie within the feasible region $[0, B_1]$ and $[0, B_2]$, respectively, which gives:

$$B_1 \geq \frac{(\lambda_S - 1)N_f}{\lambda_S(a_1 + a_2)R_0} = \frac{(\lambda_S - 1)k_1 N}{\lambda_S N R_0} = \frac{(\lambda_S - 1)k_1}{\lambda_S R_0}, \tag{88a}$$

$$B_2 \geq \frac{(\lambda_S - 1)N_f}{\lambda_S(a_1 + a_2)R_0} = \frac{(\lambda_S - 1)k_1 N}{\lambda_S N R_0} = \frac{(\lambda_S - 1)k_1}{\lambda_S R_0}. \tag{88b}$$

Combining everything we get the following conditions for the existence of the limiting MSNE in this scenario:

$$B_1 \geq \frac{(\lambda_S - 1)k_1}{\lambda_S R_0}, B_2 \geq \frac{(\lambda_S - 1)k_1}{\lambda_S R_0}. \tag{89}$$

Appendix Q
Proof of Theorem 7

We know that:

$$\begin{aligned} S_i =& \lambda B_{i,S} R_0 u' \Big[\frac{\lambda \sum_{i=1}^{N} B_{i,S} R_0}{N_f} \Big] + \\ & (B_i - B_{i,S}) R_0 u' \Big[\frac{\sum_{i=1}^{N} (B_i - B_{i,S}) R_0}{N_m} \Big]. \end{aligned} \tag{90}$$

Therefore:

$$\begin{aligned} \frac{\partial^2 S_i}{\partial B_{i,S}^2} =& {R_0}^2 \Big[\frac{2\lambda_S^2}{N_f} u''(R_S) + \frac{2}{N_m} u''(R_M) + \\ & \frac{\lambda_S^3 B_{i,S} R_0}{N_f^2} u'''(R_S) + \\ & \frac{(B_i - B_{i,S}) R_0}{N_m^2} u'''(R_M) \Big], \end{aligned} \tag{91a}$$

$$\begin{aligned} \frac{\partial^2 S_i}{\partial B_{i,S} \partial B_{j,S}} =& {R_0}^2 \Big[\frac{\lambda_S^2}{N_f} u''(R_S) + \frac{1}{N_m} u''(R_M) + \\ & \frac{\lambda_S^3 B_{i,S} R_0}{N_f^2} u'''(R_S) + \\ & \frac{(B_i - B_{i,S}) R_0}{N_m^2} u'''(R_M) \Big]. \end{aligned} \tag{91b}$$

To simplify the notation, let:

$$a = \frac{\lambda_S^2 R_0^2}{N_f} u''(R_S), b = \frac{R_0^2}{N_m} u''(R_M), \tag{92a}$$

$$c = \frac{\lambda_S^3 R_0^3}{N_f^2} u'''(R_S), d = \frac{R_0^3}{N_m^2} u'''(R_M). \tag{92b}$$

Next setting $\theta_i = 1, \forall i \in \mathcal{N}$, we can then construct the Jacobian matrix G and formulate the symmetric matrix $G_0 = G + G^T$.

$$G_{0_{ii}} = 4(a + b) + 2B_{i,S} c + 2B_{i,M} d, \tag{93a}$$

$$G_{0_{ij}} = 2(a + b) + (B_{i,S} + B_{j,S})c + (B_{i,M} + B_{j,M})d. \tag{93b}$$

Letting $x = [k_1 \quad k_2 \cdots k_N]^T$, we have:

$$\begin{aligned} x^T G_0 x =& (a + b) \big[4 \sum_{i=1}^{N} k_i^2 + 2 \sum_{i=1}^{N} k_i \sum_{j \neq i}^{N} k_j \big] + \\ & c \big[\sum_{i=1}^{N} B_{i,S} (2k_i^2 + 2k_i \sum_{j \neq i} k_j) \big] + \\ & d \big[\sum_{i=1}^{N} B_{i,M} (2k_i^2 + 2k_i \sum_{j \neq i} k_j) \big]. \end{aligned} \tag{94}$$

Denoting

$$P = 4 \sum_{i=1}^{N} k_i^2 + 2 \sum_{i=1}^{N} k_i \sum_{j \neq i}^{N} k_j, \tag{95a}$$

$$L = \max\{2k_i^2 + 2k_i \sum_{j \neq i} k_j\}, \tag{95b}$$

then we have:

$$\begin{aligned} x^T G_0 x \leq& P(a + b) + L(B_{1,S} + B_{2,S} + \cdots + B_{N,S})c + \\ & L(B_{1,M} + B_{2,M} + \cdots + B_{N,M})d \qquad (96a) \\ =& P(a + b) + L(B_S c + B_M d). \qquad (96b) \end{aligned}$$

Based on our assumptions, $xu'''(x) + 2u''(x) < 0$, and we have:

$$\begin{aligned} B_S c + B_M d =& \frac{\lambda_S^3 B_S R_0^3}{N_f^2} u'''(R_S) + \frac{B_M R_0^3}{N_m^2} u'''(R_M) \qquad (97a) \\ =& \frac{\lambda_S^2 R_0^2}{N_f} R_S u'''(R_S) + \frac{R_0^2}{N_m} R_M u'''(R_M) \qquad (97b) \\ <& -\frac{2\lambda_S^2 R_0^2}{N_f} u''(R_S) - \frac{2R_0^2}{N_m} u''(R_M) \qquad (97c) \\ =& -2(a + b). \qquad (97d) \end{aligned}$$

Therefore, we only need to compare P and $2L$:

$$P = 4 \sum_{i=1}^{N} k_i^2 + 2 \sum_{i=1}^{N} k_i \sum_{j \neq i}^{N} k_j = 4 \sum_{i=1}^{N} k_i^2 + 4k_i k_{j\,(i \neq j)}, \tag{98a}$$

$$2L = \max\{4k_i^2 + 4k_i \sum_{j \neq i} k_j\} = \max\{4k_i^2 + 4k_i k_{j\,(i \neq j)}\}. \tag{98b}$$

Obviously $P > 2L$. Therefore:

$$x^T G_0 x < (P - 2L)(a + b) < 0, \tag{99}$$

which proves that G_0 is negative definite.

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