The Impact of Observation and Action Errors on Informational Cascades

Tho Ngoc Le, Vijay G. Subramanian, Randy Berry

***Abstract*— In models of observational learning among Bayesian agents *informational cascades* can result, in which agents ignore their private information and blindly follow the actions of other agents. This paper considers the impacts of two types of errors in such models: action errors, where agents occasionally choose sub-optimal actions and observation errors, where agents observe the action of another agent incorrectly. We investigate and compare the impact of these two types of errors on the agents' welfare and probability of wrong herding. Using a Markov chain model, we derive the net payoff of each agent as a function of his private signal quality and the error rates. A main result of this analysis is that in certain cases, increasing the observation error rate can lead to higher welfare when the number of agents is large.**

I. Introduction

Consider a recommendation system where agents sequentially decide whether to buy an item, for which they have some prior knowledge of its quality/utility. Later agents can potentially benefit from the information obtained by observing their predecessors' choices. Such systems, however, can lead to herding. *Herding* or an *informational cascade* occurs when it is optimal for the agents to ignore their own signals and follow the actions of others. In addition to the possibility of herding to the wrong conclusion, an informational cascade results in a loss of information about the private signals held by all the agents following the onset of herding.

The study of herding was initiated in the seminal papers [6], [7], and [8], which cast this in an observational Bayesian learning framework. In such models, each individual has some prior knowledge or signal about some payoff-relevant state of the world generated according to a commonly known probability distribution. Individuals make decisions sequentially and observe exactly the decisions made by all previous agents. Given these observations and their own signals, agents are assumed to be Bayesian rational, i.e., they choose the action that reflects their posterior belief about the state of the world. Assuming bounded private signals, these assumptions lead to a positive probability of herding toward the wrong choice.

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In this paper we consider a similar model as in [6]-[8] except we introduce two sources of errors: *action errors* and *observation errors*. Action errors capture the more realistic assumption that an agent's action may not be perfectly rational. For example, even highly sophisticated decision makers such as investors ([18]) and consumers ([17]) can make mistakes in choosing binary actions (investing in stocks or purchasing goods). Indeed, there is a long history of introducing noise as a way of relaxing the assumption of perfect rationality (see e.g. [5], [9]) resulting in agents having so-called "noisy best responses." We follow this approach here, e.g. we assume that all agents form exact posterior beliefs but then choose the incorrect action with a given probability. Further this probability is known to all other agents and is assumed to be the same for all agents.

In fact, the reasons underlying human irrationality have been well argued in [13] where the authors presented the numerous cognitive deficiencies that could account for human systematic deviation from the perceived normative behaviors. Further, from the viewpoint of Prospect Theory ([3]), the action error can also be argued to be a consequence of inconsistency in preferences when agents tend to act differently under the possibilities of gains (facing a correct herding) and losses (being in a wrong herd). Finally, such an optimality deviation has also been used for equilibrium selection, in particular, for trembling-hand perfection equilibrium ([2]).

We also consider observation errors, namely each agent's observations are not a perfect replica of the previous agents actions. More precisely, we consider a model where each agents actions are recorded for all subsequent agents to see, but this record is subject to error, again with the statistics of the error process known to all agents. As an example, this could model a setting where agents are asked to report their decisions on a web site and agents occasionally misreport. Alternatively, this could result from either strategic agents or a social planner who would manipulate the actions history toward their benefits.

In summary, our model assumes that agents can make errors in choosing their actions resulting in action errors, and, in addition, the information history of past actions can also be in error leading to observation errors. The objective then is to study and compare the effect of these two types of errors in the type of Bayesian learning model considered in [6]-[8]. To this end, we develop a Markov chain model that we use to analyze the probability of herding and the expected pay-off of each agent as a function of the error-rates and signal quality. Our results demonstrate a counter-intuitive phenomenon: for certain parameters, the agents' total payoff can be increased by increasing the errors rate in the observation process. The extent of this phenomenon and the amount of noise to be added depend on the agents' signal quality and the total amount of noise (function of action and observation errors to be clearly specified later on) already present in the model.

Other than the addition of errors, we retain the rest of the model from [6]-[8]: individuals take actions sequentially, agents observe all past actions and each agent's private signal has a bounded likelihood ratio between the two alternatives. More general models of observational learning have been studied in the literature without errors (e.g. [10], [11], [15], [16]). For Bayesian learning, the work in [11] shows that allowing for unbounded private signals can prevent herding from occurring and the work in [15] considers more general information structures as well as allows for unbounded signals.

Another strand of work related to our model of action errors are models that allow for heterogeneous preferences among the agents (e.g. [11], [14], [19]). In these models agents have different types, which are private information. As argued in [11], from the view of other agents, a set of agents having a different type is equivalent to some agents randomly choosing their actions as in our model. In this case, in [11] it shown that herding may still occur which is consistent with our model. Further, compared to [11], we give a complete characterization of the impact of such errors and also allow for observation errors.

This paper is organized as follows. In Section II we first develop a version of the model in [6] that allows for both action and observation errors. We study the effect of the two types of errors in Section III and model this as a Markov chain. In Section IV, we study how the probability of herding and the agent welfare vary as a function of the noise level and signal qualities. We conclude in Section V.

II. MODEL

We consider a model similar to [6] in which there is a countable population of agents, indexed $i = 1, 2, \ldots$ with the index reflecting both the time and order of actions of the agents. Each agent i has an action choice A_i of saying either Yes (Y) or No (N) to a new item. The true value (V) of the item can be either 0 (bad) or 1 (good); both possibilities are assumed to be equally likely. The agents are Bayes-rational utility maximizers whose payoff structure is based on the agent's choice of action and the true value of the item. If an agent chooses N, his payoff is 0. On the other hand, if he chooses Y, he faces a cost of $C = 1/2$ and two possibilities depending on the true value of the item: his gain is 0 if $V = 0$ and 1 if $V = 1$. Thus, the *ex-ante* payoff of each agent is $E[V] - C = 0$. To reflect the agents' prior knowledge about the true value of the item, we assume that each agent i receives a private signal S_i through a binary symmetric channel (BSC) with crossover probability $1-p$, where $1/2 < p < 1$. (See Fig. 1.) Thus, the private signals are informative, but not revealing. Note also that the likelihood ratios of the value of the item based on the private signal remains bounded. We further assume that each agent i makes a one-time action A_i based on his own private signal S_i and the observations $O_1, \ldots, O_{i-1}$ of all previous agents' actions $A_1, \ldots, A_{i-1}$.

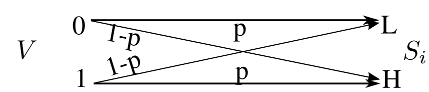

Figure 1: The BSC over which agents receive signals.

It was shown in [6] and [12] that when agents are rational and observations are perfect (i.e., $O_i = A_i$), this model exhibits a *herding* phenomenon. *Herding* happens when an agent chooses to follow his predecessor's action regardless of his own private signal. Here, we instead consider a noisy model where the noise appears as a result of two types of error:

1) Every agent takes a non-optimal action with probability ϵ_1, where $\epsilon_1 \in (0, 1/2)$. We call this *action error*;
2) Later agents' observations are noisy versions of their predecessors' actions. We call this *observation error*. For simplicity, this error is modelled by passing every action A_i through another BSC with crossover probability $\epsilon_2 \in (0, 1/2)$.

$$O_i = \begin{cases} A_i, & \text{with probability } 1 - \epsilon_2. \\ \overline{A_i}, & \text{with probability } \epsilon_2, \end{cases} \quad (1)$$

We further assume that each agent reports his action to a public database which is available to all successors. The added noises reduce the dependence of every agent's decision on the predecessors' choices and drives him toward using his own signal.

III. Herding and error thresholds

The first agent always follows his private signal since no observation history is available. Starting from the second agent, every agent i considers his private signal S_i and the observations $O_1, ..., O_{i-1}$. Let the information set of agent i be $I_i = \{S_i, O_1, ..., O_{i-1}\}$. Based on I_i, agent i will update his posterior probability denoted as $\gamma_{i,I_i} = Pr[V = 1|I_i]$ using Bayes' formula. If this posterior probability is greater than the cost C, the agent will choose Y. If γ_{i,I_i} is less than C, the agent i will declare N. Finally, if γ_{i,I_i} equals the cost, then agent i follows his private signal.[1]

In contrast to the model in [6], where the second agent has 50% of the chance creating a herd, in our model he always follows his own signal. However, depending on the action error and the observation error, this will not be the case starting from the third agent. Let us define ϵ as the total amount of error introduced in the information set of the agents: with probability $1-\epsilon$, an agent sees the actual optimal action of each of his predecessors. It follows from the description in Section II that $\epsilon = \epsilon_1(1-\epsilon_2)+(1-\epsilon_1)\epsilon_2$. As the total amount of error ϵ introduced into the model is increased, the index of the first agent who can herd also increases as shown in the following lemma.

Lemma 1. *An agent $k \geq 3$ will never herd if the noise level ϵ satisfies $\epsilon \geq \epsilon^*(k,p)$, where:*

$$\epsilon^*(k,p) = \frac{1-(\frac{1-p}{p})^{\frac{k-2}{k-1}}}{1-(\frac{1-p}{p})^{\frac{k-2}{k-1}}+(\frac{1-p}{p})^{\frac{-1}{k-1}}-(\frac{1-p}{p})}. \quad (2)$$

In other words, if ϵ satisfies the condition in this lemma, there is no possible information set for agent k, under which it would ignore its own signal. The proof of this follows from direct calculation of γ_{i,I_i}. Fig. 2 shows the thresholds $\epsilon^*(k,p)$ for different values of k and $p \in (1/2, 1)$.

[1] This differs from [6], where it is assumed that indifferent agents randomly choose one action. Our assumption simplifies the analysis but does not qualitatively change the conclusions.

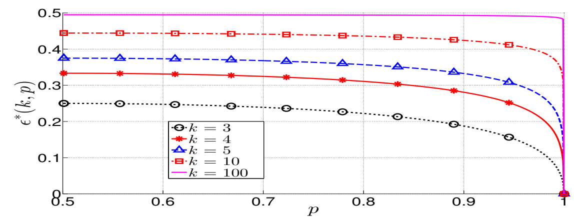

Figure 2: Thresholds for ϵ

From (2) we can obtain some useful insights about the properties of the threshold $\epsilon^*(k,p)$. First, $\epsilon^*(k,p)$ is an increasing function of k, and as $k \to \infty$, $\epsilon^*(k,p) \to 1/2$ for all values of signal quality $p \in (1/2, 1)$. Later agents have a higher likelihood of herding and such effects can be countered if the whole model is noisier. In the limit, if $\epsilon = 0.5$, no information is passed through. In this case, agents only use their own signals and herding is prevented. However, no learning occurs either, and the *ex-post* payoff of each agent remains $\frac{2p-1}{4}$. Secondly, $\epsilon^*(k,p)$ is a decreasing function in $p \in (1/2, 1)$; as $p \to 1$, $\epsilon^*(k,p) \to 0$. This agrees with the intuition that the more accurate the private signal, the less likely it is for herding to occur in the "wrong" direction. Notice also that the threshold curves are relatively flat for a wide interval of p and only drop quickly when p is sufficiently close to 1. This means that even if the private signal quality is very high, with an intermediate level of noise herding may still occur for most agents.

A. Consequences of noisy observations

In this section we outline some basic properties of herding with action errors and observation errors. These naturally extend properties for the noiseless case shown in [6], [12] and so we omit detailed derivations.

Property 1. *Until herding occurs, each agent's Bayesian update depends only on their private signal and the difference in the number of Y's and N's in the observation history.*

In other words, the difference in the number of Y's and N's is a sufficient statistic for the observation history; we denote this quantity by $\#Y's - \#N's$. This follows from the symmetry of the signal quality and the two types of noises, which enables each agent to "cancel out" opposite observations.

Property 2. *Once herding happens, it lasts forever.*

The reason for this phenomenon is that when herding starts, agents stop using their private signals and thus provide no more information to their successors. The successors are left in the same situation as the first agent who started herding and thus have the same optimal action choice.

Property 3. *Assume that $\epsilon^*(k,p) \leq \epsilon < \epsilon^*(k+1,p)$. At any point in the process, if $|\#Y's - \#N's| \geq k$, herding will start and all subsequent observations will be ignored by the subsequent agents.*

This is shown by using Properties 1 and 2. Property 3 helps establish a simple finite-state birth-death Markov chain for our model as presented in the next section. Note also that herding occurs with probability one in our model and with non-zero probability this herding will be in in the "wrong" direction.

B. Markov analysis of herding

By the symmetry of the model, first consider the case $V = 1$. From the previous section, for each agent who has not herded, the observations history can be summarized by $(\#Y's - \#N's)$. Thus, viewing each agent as a time-epoch, we can consider the agent's observation as a state of a finite-state discrete-time Markov chain (DTMC). Each state i represents values of $(\#Y's - \#N's)$ that an arbitrary agent may see before making his decision. Note that the first agent starts at state 0 since no observation history is available.

For the rest of the paper, assume $\epsilon^*(k,p) \leq \epsilon < \epsilon^*(k+1,p)$, so that an agent will not herd unless he observes that $|\#Y's - \#N's| = k$. Since herding lasts forever once it starts, the state space of this Markov Chain is $\{-k, -k+1, ..., 0, ..., k-1, k\}$ with states $\pm k$ being absorbing. Thus, the event that an agent n faces the herd toward Y (or N) actions is equivalent to the event that the state at the time $n-1$ is k (or $-k$). The probability of moving one step to the right is the probability that one more Y is added to the observation history, i.e., $a = Pr[O_i = Y|V = 1] = (1-\epsilon)p + \epsilon(1-p) > 1/2$. Likewise, the probability of moving one-step to the left is $1-a$. Hence, this Markov chain is a simple random walk with a drift to the right as shown in Fig. 3.

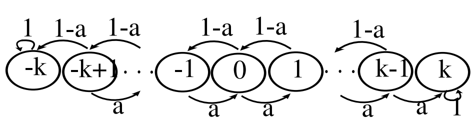

Figure 3: Transition diagram of the random walk when V=1.

The state transition matrix of this MC is given by

$$Q = \begin{bmatrix} 1 & 0 & 0 & ... & 0 & 0 & 0 \\ 1-a & 0 & a & ... & 0 & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & ... & 1-a & 0 & a \\ 0 & 0 & 0 & ... & 0 & 0 & 1 \end{bmatrix} \quad (3)$$

Since $a > 0.5$, Q is a row stochastic matrix with a drift to the right. We will use methods developed in [1] to calculate the probability of being at either absorbing state at any given time; since the agent indices correspond to the time index, this yields the probability of herding of each agent. Assume that the process starts at state i. Let $u^*_{i,n}, v^*_{i,n}$ be the probabilities of being at the left wall, $-k$, and the right wall, k, at the n^{th} step, respectively. Let $u_{i,n}, v_{i,n}$ be the probabilities of hitting the left wall and the right wall for the first time at the n^{th} step, respectively. Note that if $n-i-k$ is an odd number, the chain cannot be at either wall for the first time, thus $u_{i,n} = v_{i,n} = 0$. Therefore, the absorption probabilities at steps n and $n-1$ are identical, i.e., $v^*_{i,n} = v^*_{i,n-1}$ and $u^*_{i,n} = u^*_{i,n-1}$. Moreover, as agent 1 starts at step 0, agent $n+1$ cannot herd if $n \leq k-1$, i.e., $u^*_{i,n} = v^*_{i,n} = 0$ for $1 \leq n \leq k-1$. For $n \geq k$, the probabilities of agent $n+1$ herding the wrong and correct way are, respectively:

$$u^*_{0,n} = \sum_{\substack{m=k \\ (m-k) even}}^{n} u^*_{-k,n-m} u_{0,m} = \sum_{\substack{m=k \\ (m-k) even}}^{n} u_{0,m}, \quad (4)$$

$$v^*_{0,n} = \sum_{\substack{m=k \\ (m-k) even}}^{n} v^*_{k,n-m} v_{0,m} = \sum_{\substack{m=k \\ (m-k) even}}^{n} v_{0,m}, \quad (5)$$

since $u^*_{-k,n-m} = v^*_{k,n-m} = 1$ (once agent m is the first one to herd, the subsequent agents $m+1, ..., n$ will herd with probability 1). The next lemma gives explicit expressions for the terms on the right-hand side in (4) and (5).

Lemma 2.

$$u_{0,n} = \begin{cases} 0, & n-k \textit{ odd}, \\ \frac{1}{k} 2^n a^{\frac{n-k}{2}} (1-a)^{\frac{n+k}{2}} A_{k,n}, & n-k \textit{ even}, \end{cases} \quad (6)$$

$$v_{0,n} = \begin{cases} 0, & n-k \textit{ odd}, \\ \frac{1}{k} 2^n a^{\frac{n+k}{2}} (1-a)^{\frac{n-k}{2}} A_{k,n}, & n-k \textit{ even}, \end{cases} \quad (7)$$

where

$$A_{k,n} = \sum_{\substack{\nu=1 \\ \nu \ odd}}^{\nu<k} \cos^{n-1}\left(\frac{\nu\pi}{2k}\right) \sin\left(\frac{\nu\pi}{2k}\right) (-1)^{\frac{\nu-1}{2}}. \quad (8)$$

Proof See the Appendix. □

By symmetry, for the case $V = 0$, the probabilities that agent n is the first one to reach the right and the left walls are $\tilde{v}_{0,n} = u_{0,n}, \tilde{u}_{0,n} = v_{0,n}$, respectively. Thus $\tilde{v}^*_{0,n} = u^*_{0,n}, \tilde{u}^*_{0,n} = v^*_{0,n}$.

IV. Effect of action errors and observation errors

In this section we use the results of Lemma 1 and 2 to analyze the effect of varying the error rates on both the probability of herding and the welfare of each agent.

A. Herding probabilities

Lemma 2 shows that the probabilities of wrong and correct herding depend on k and a. From Lemma 1 and since $a = (1-\epsilon)p + \epsilon(1-p)$, for a fixed signal quality p these probabilities are ultimately determined by the total error $\epsilon = \epsilon_1(1-\epsilon_2) + (1-\epsilon_1)\epsilon_2$. Therefore, for a fixed total error ϵ, varying either the action error ϵ_1 or ϵ_2 yields the same effect on the epoch where herding starts and on the probability that herding happens for each agent n.

The next theorem characterizes the effect of varying the total error ϵ on the probability of wrong and correct herding for an arbitrary agent.

Theorem 1. *For ϵ in between any two consecutive thresholds $\epsilon^*(k,p)$ and $\epsilon^*(k+1,p)$, for any agent i:*

1) *The probability of wrong herding, $u^*_{0,i}$, increases with ϵ.*
2) *The probability of correct herding, $v^*_{0,i}$, decreases with ϵ.*

Proof Consider $\epsilon^*(k,p) < \epsilon^{'} < \epsilon^{''} < \epsilon^*(k+1,p)$, and let $\{Z_n^{'}\}_{n\geq 0}$ and $\{Z_n^{''}\}_{n\geq 0}$ be two DTMCs on the same state space $S = \{-k, -k+1, ..., 0, ..., k-1, k\}$ corresponding to $\epsilon^{'}$ and $\epsilon^{''}$, respectively. We use the following concept of *stochastic ordering* for comparing these two DTMCs [4]:

Definition 1. *Let X and Y be two discrete random variables taking values on the same set S and let x and y be their corresponding probability distribution vectors. $X(x)$ is said to be larger than $Y(y)$ in stochastic ordering, denoted by $X \geq_{st} Y (x \geq_{st} y)$, if*

$$\sum_{i\geq j} x_i \geq \sum_{i\geq j} y_i, \text{ for all } i,j \in S. \tag{9}$$

Definition 2. *The chain $\{Z_n^{'}\}$ is said to be larger than the chain $\{Z_n^{''}\}$ in stochastic ordering, denoted by $\{Z_n^{'}\} \geq_{st} \{Z_n^{''}\}$, if*

$$Z_n^{'} \geq_{st} Z_n^{''}, \text{ for all } n \geq 0. \tag{10}$$

Definition 3. *A transition matrix Q is said to be stochastically increasing if for all $i, i-1 \in S$:*

$$Q_i \geq_{st} Q_{i-1} \tag{11}$$

where Q_i denotes the i^{th} row of Q.

The proof continues by noting that the corresponding transition probabilities to the right of the two chains satisfy $a^{'} > a^{''} > 0.5$. Thus, using (3), we have $Q_i^{'} \geq_{st} Q_i^{''}$ for all $i \in S$. Moreover, these matrices are stochastically increasing, and both $\{Z_n^{'}\}$ and $\{Z_n^{''}\}$ start from the same initial state 0. Therefore, by Theorem 4.2.5a and equation (4.2.16') in [4], we have $\{Z_n^{'}\} \geq_{st} \{Z_n^{''}\}$. Let $Z_n^{'}$ and $Z_n^{''}$ be the corresponding states at an arbitrary time n. By Definition 2, we have $Z_n^{'} \geq_{st} Z_n^{''}$. Let $z^{'}$ and $z^{''}$ be the corresponding probability distribution vectors at time n. Then using Definition 1, let $j = k$ we have $v^{*'}_{0,n} = z_k^{'} \geq z_k^{''} = v^{*''}_{0,n}$. Similarly, letting $j = -k+1$ yields:

$$\sum_{i\geq -k+1} z_i^{'} \geq \sum_{i\geq -k+1} z_i^{''}$$

so that $u^{*'}_{0,n} \leq u^{*''}_{0,n}$. We will next use a coupling argument to prove that equality does not hold for $n > k$, i.e., that $v^{*'}_{0,n}$ is strictly greater than $v^{*''}_{0,n}$ for $n > k$. Using Proposition 1.10.4 in [4], there exist two random variables Z^1_{n-1} and Z^2_{n-1} on a common probability space such that $\mathbb{P}(Z^{'}_{n-1} = k) = \mathbb{P}(Z^1_{n-1} = k)$, $\mathbb{P}(Z^{''}_{n-1} = k) = \mathbb{P}(Z^2_{n-1} = k)$ and the stochastic order also holds for Z^1_{n-1} and Z^2_{n-1} almost surely, i.e.

$$Z^1_{n-1} \geq_{st} Z^2_{n-1} \ a.s. \tag{12}$$

Define two independent random variables A_n^1 and B_n that are also independent of Z^1_{n-1} and Z^2_{n-1} by:

$$A_n^1 = \begin{cases} 1, & \text{with probability } a^{'}, \\ -1, & \text{with probability } 1-a^{'}. \end{cases} \tag{13}$$

$$B_n = \begin{cases} 0, & \text{with probability } \frac{a^{''}}{a^{'}}, \\ -2, & \text{with probability } 1-\frac{a^{''}}{a^{'}}. \end{cases} \tag{14}$$

Next, define the random variable A_n^2 by:

$$A_n^2 = A_n^1 + \mathbb{1}_{A_n^1=1} B^n. \tag{15}$$

By (13)-(15), we have $A_n^1 \geq A_n^2$ and A_n^2 has the following distribution:

$$A_n^2 = \begin{cases} 1, & \text{with probability } a^{''}, \\ -1, & \text{with probability } 1-a^{''}. \end{cases} \tag{16}$$

Using the above, define Z_n^j for $j = 1,2$ as:

$$Z_n^j = \begin{cases} Z^j_{n-1}, & \text{if } |Z^j_{n-1}| = k, \\ Z^j_{n-1} + A^j_{n-1}, & \text{otherwise.} \end{cases} \tag{17}$$

Thus, Z_n^1 and Z_n^2 have the same distributions as $Z_n^{'}$ and $Z_n^{''}$, respectively. Now, $\mathbb{P}r(Z_n^2 = k)$ can be written as:

$$\mathbb{P}(Z^2_{n-1} = k) + \mathbb{P}(Z^2_{n-1} = k-1, A_n^2 = 1). \tag{18}$$

Since $Z_{n-1}^1 \geq_{st} Z_{n-1}^2$, $Z_{n-1}^2 = k$ implies $Z_{n-1}^1 = k$. Moreover, $Z_{n-1}^2 = k-1$ also implies $Z_{n-1}^1 = k-1$ or $Z_{n-1}^1 = k$. Thus (18) can be decomposed as:

$$\begin{aligned}&\mathbb{P}(Z_{n-1}^2 = k, Z_{n-1}^1 = k) \quad (19)\\ &+\mathbb{P}(Z_{n-1}^2 = k-1, Z_{n-1}^1 = k, A_n^2 = 1)\\ &+\mathbb{P}(Z_{n-1}^2 = k-1, Z_{n-1}^1 = k-1, A_n^2 = 1).\end{aligned}$$

Since $A_n^2 = 1$ implies $A_n^1 = 1$, (19) is smaller than:

$$\begin{aligned}&\mathbb{P}(Z_{n-1}^2 = k, Z_{n-1}^1 = k) \quad (20)\\ &+\mathbb{P}(Z_{n-1}^2 = k-1, Z_{n-1}^1 = k, A_n^1 = 1)\\ &+\mathbb{P}(Z_{n-1}^2 = k-1, Z_{n-1}^1 = k-1, A_n^1 = 1).\end{aligned}$$

Now, note that:

$$\begin{aligned}&\mathbb{P}(Z_{n-1}^2 = k-1, Z_{n-1}^1 = k-1, A_n^1 = 1)\\ &< \mathbb{P}(Z_{n-1}^1 = k-1, A_n^1 = 1),\\ &\mathbb{P}(Z_{n-1}^2 = k, Z_{n-1}^1 = k)\\ &+\mathbb{P}(Z_{n-1}^2 = k-1, Z_{n-1}^1 = k, A_n^1 = 1)\\ &< \mathbb{P}(Z_{n-1}^1 = k) \text{ and}\\ &\mathbb{P}(Z_n^2 = k) < \mathbb{P}Z_{n-1}^1 = k) \quad (21)\\ &+\mathbb{P}(Z_{n-1}^1 = k-1, A_n^1 = 1) = \mathbb{P}(Z_n^1 = k).\end{aligned}$$

Thus, this implies $\mathbb{P}(Z_n^{''} = k) < \mathbb{P}(Z_n^1 = k)$, i.e. $v_{0,n}^{*''} < v_{0,n}^{*'}$. Similarly, $u_{0,n}^{*''} > u_{0,n}^{*'}$. Since Lemma 2 shows that $u_{0,n}^*$ and $v_{0,n}^*$ are continuous functions of ϵ, this completes the proof. □

As a demonstration of Theorem 1, Fig. 4 shows the probability of wrong herding when the signal quality is high, $p = 0.99$, agents $i = 5, 10$ and 100. Notice that for any agent i, Lemma 1 shows that there exists ϵ close enough to 0.5 that yields $k \geq i$. Herding, thus, does not happen and hence the probability of wrong herding will go to zero for large enough ϵ, as shown in Fig. 4. Note also that for each agent i, as ϵ increases the probability of wrong herding discontinuously decreases at a finite number of points. These points correspond exactly to the values $\epsilon^*(k, p)$ for different choices of k.

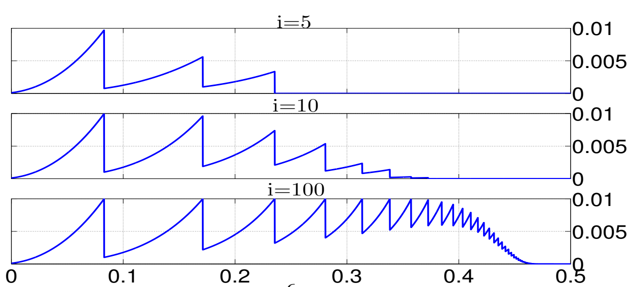

Figure 4: Probability of wrong herding as a function of ϵ for agent i and $p = 0.99$

B. Agent Welfare

Let π_i be the payoff or welfare of agent i. From Section II we have $\pi_i = 0$ if $A_i = N$, while if $A_i = Y$, π_i is either $1/2$ or $-1/2$ corresponding to $V = 1$ or $V = 0$, respectively. All agents i from 1 to k use their own signals, thus they all have the same welfare given by:

$$\begin{aligned}E[\pi_i] &= \frac{1}{4}\{P[A_i = Y|V=1] - P[A_i = Y|V=0]\}\\ &= (1-2\epsilon_1)\frac{2p-1}{4} = \frac{2a_1-1}{4} \triangleq F. \quad (22)\end{aligned}$$

where $a_1 = (1-\epsilon_1)p + \epsilon_1(1-p)$.

For agents $i \geq k+1$:

$$\begin{aligned}&E[\pi_i] = \frac{1}{4}\{P[A_i = Y|V=1] - P[A_i = Y|V=0]\}\\ &= \frac{2a_1-1}{4} + \frac{1}{2}\left[(1-a_1-\epsilon_1)v_{0,i-1}^* - (a_1-\epsilon_1)u_{0,i-1}^*\right]\\ &= F + (1-2\epsilon_1)\left[\frac{1-p}{2}v_{0,i-1}^* - \frac{p}{2}u_{0,i-1}^*\right]. \quad (23)\end{aligned}$$

Thus, for a fixed total error ϵ, (22) and (23) suggest that the action error ϵ_1 results in the welfare of every agent being reduced by a factor of $1-2\epsilon_1$. The following theorem shows some properties of the agents' welfare.

Theorem 2. *With the same signal quality p and k satisfying $\epsilon^*(k,p) \leq \epsilon < \epsilon^*(k+1,p)$, we have:*

1) *The expected welfare for each agent is at least equal to the expected welfare of his predecessors. Thus $E[\pi_i] \geq F$ and is non-decreasing in i.*
2) *$\lim_{i\to\infty} E[\pi_i]$ exists and equals:*

$$\Pi(\epsilon_1,\epsilon_2) = F + \frac{(1-2\epsilon_1)}{2}\left[\frac{1}{1+\left(\frac{1-a}{a}\right)^k} - p\right]. \quad (24)$$

3) *The expected welfare of every agent i, $E[\pi_i]$, decreases continuously as ϵ increases over the range where k is fixed so that:*

$$\lim_{\epsilon\downarrow\epsilon^*(k,p)} E[\pi_i] > E[\pi_i] > \lim_{\epsilon\uparrow\epsilon^*(k+1,p)} E[\pi_i] = F. \quad (25)$$

Proof 1) When herding happens, every user takes the same action and thus achieves the same expected welfare. Thus, we are left to show $E[\pi_i] \geq F$ for all $i \geq k+1$. Using (23) and the form of $v_{0,i-1}^*$, $u_{0,i-1}^*$, we only need to show:

$$(1-p)a^{\frac{j+k}{2}}(1-a)^{\frac{j-k}{2}} - pa^{\frac{j-k}{2}}(1-a)^{\frac{j+k}{2}} \geq 0 \quad (26)$$

which can be seen by noting that $\epsilon^*(k,p) \leq \epsilon < \epsilon^*(k+1,p)$ leads to $0 < \left(\frac{1-a}{a}\right)^k < \frac{1-p}{p} < 1$.

2) Let $V_0(s)$ and $U_0(s)$ be the probability generating function for the first hitting time of state k and $-k$, respectively. Using these, the limiting welfare can be written as:

$$\Pi(\epsilon_1, \epsilon_2) - F = \frac{1 - 2\epsilon_1}{2} \left[(1-p)V_0(1) - pU_0(1) \right]. \tag{27}$$

Expressions for these generating functions are given in equations (35) and (36) in the Appendix; evaluating these at $s = 1$ yields (24).

3) For a fixed p, (22) shows that F decreases in ϵ. The proof follows by using (23) and Theorem 1. □

Assume that the action error ϵ_1 and the observation error ϵ_2 are fixed. Suppose a social planner is allowed to randomly change the history of actions with probability ϵ_3. Thus, this introduces an effect that is equivalent to increasing the observation error to $\tilde{\epsilon}_2 = (1-\epsilon_2)\epsilon_3 + \epsilon_2(1-\epsilon_3)$. Fig. 5 and 6 show an example when $\epsilon_1 = 0.05$, $\epsilon_2 = 0.1$, adding more observation error (i.e. letting $\epsilon_3 > 0$) benefits the expected agent's welfare at infinity, $\Pi(\epsilon_1, \tilde{\epsilon}_2)$, for both high and low signal quality.

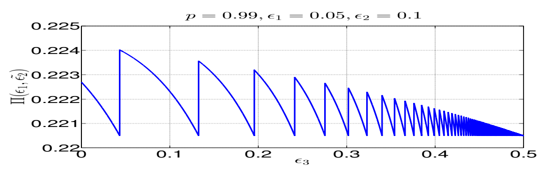

Figure 5: The opportunity to increase expected welfare at infinity when signal quality is high

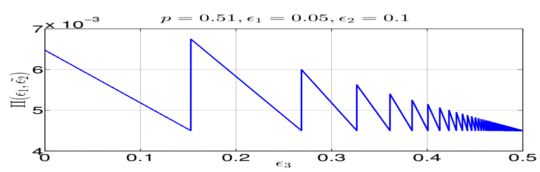

Figure 6: The opportunity to increase expected welfare at infinity when signal quality is low

By part 1) of Theorem 2, this also maximizes the (Cesàro) average social welfare of the entire population. We formalize this phenomenon in the following theorem.

Theorem 3. *Assume that $\epsilon^*(k,p) \le \epsilon < \epsilon^*(k+1,p)$:*

1) *The asymptotic social welfare is maximized at either $\epsilon_3 = 0$ or $\epsilon_3 = \frac{\epsilon^*(k+1,p)-\epsilon}{1-2\epsilon}$, and*
2) *The latter case happens when*

$$\epsilon^*(k+1,p) > \epsilon > \frac{1}{1-2p} \left[\frac{1}{1 + (\frac{1-a}{a})^{(k+1)/k}} - p \right], \tag{28}$$

where $a = [1 - \epsilon^*(k+1,p)]\, p + \epsilon^*(k+1,p)(1-p)$.

Proof 1) The proof follows by part 3 of Theorem 2.
2) We need to find ϵ_3 such that $\Pi(\epsilon_1, \epsilon_2) < \Pi(\epsilon_1, \tilde{\epsilon}_2)$. Using (24), we obtain the lower bound as in (28). □

Note that when it is optimal for the social planner to add $\epsilon_3 > 0$, the welfare of every individual agent is not necessarily improved. This is because at the beginning of each new threshold epoch, the welfare of an agent is less than that of his successor. Thus by increasing ϵ_3, the welfares of the first few agents will be decreased while this increases the welfare for all successive agents. This effect is shown in Fig. 7 and 8 with the same values $\epsilon_1 = 0.05$ and $\epsilon_2 = 0.1$. For high signal quality, increasing ϵ_3 is beneficial for all agents $i = 5, 10$ and 100; whereas for low signal quality, this will increase the welfare of agent $i = 100$ and decrease the welfare of agents $i = 5, 10$.

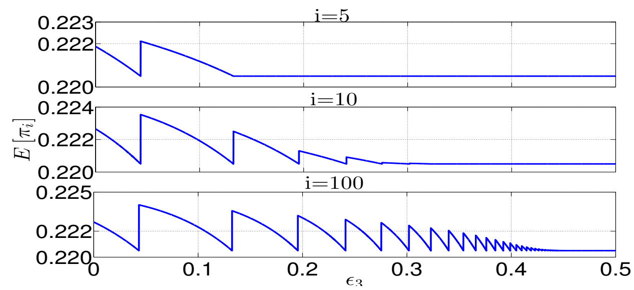

Figure 7: Expected welfare for agent i when signal quality is high

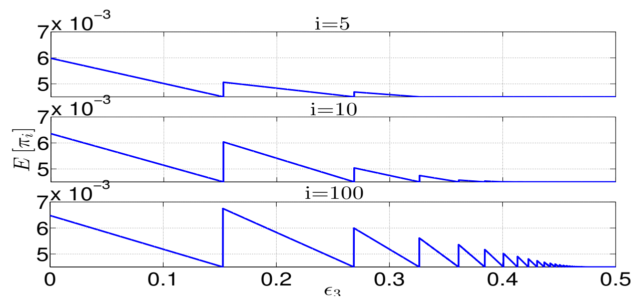

Figure 8: Expected welfare for agent i when signal quality is low

V. Conclusions and Future Work

This paper studied the effect of two types of error in a simple Bayesian information cascade: action error and observation error. By assuming that the agents occasionally make mistakes in choosing their actions, and the history of actions has errors, the model is investigated using a Markov-chain-based analysis. We determined the probabilities of herding for an arbitrary agent and used these to calculate the agents' welfare based on the given signal quality and the two types of error. Our main result shows that for certain ranges of parameters, adding a controlled amount of observation error always increases the total welfare when the society is large. In future work we plan on generalizing to heterogeneous agents with the heterogeneity in private signals, action and observation errors and allowing each agent to only observe subsets of past agents' actions prior to taking actions.

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VI. APPENDIX

A. Proof of Lemma 2

The proof follows using techniques from [1]. Let $\tau_{-k,i}$ and $\tau_{k,i}$ be random variables denoting the first time the Markov chain hits the absorbing states $-k$ and k, respectively, starting from state i. Let $U_i(s), V_i(s)$ be the corresponding probability generating functions. We have:

$$u_{i,n} = P\left[\tau_{-k,i} = n\right], \, v_{i,n} = P\left[\tau_{k,i} = n\right], \quad (29)$$

$$U_i(s) = E[s^{\tau_{-k,i}}] = \sum_{n=0}^{\infty} u_{i,n} s^n, \quad (30)$$

$$V_i(s) = E[s^{\tau_{k,i}}] = \sum_{n=0}^{\infty} v_{i,n} s^n. \quad (31)$$

With probabilities a and $1-a$, respectively, the state one step after state i is $i+1$ or $i-1$. Thus we obtain the following the difference equations:

$$U_i(s) = asU_{i+1}(s) + (1-a)sU_{i-1}(s), \quad (32)$$

$$V_i(s) = (1-a)sV_{i+1}(s) + asV_{i-1}(s), \quad (33)$$

where $-k < i < k$, with the boundary conditions:

$$U_{-k}(s) = 1, U_k(s) = 0, V_{-k}(s) = 0, V_k(s) = 1. \quad (34)$$

The solutions to the above equations are:

$$U_i(s) = \frac{\lambda_1^{i+k}(s)\lambda_2^{2k}(s) - \lambda_1^{2k}(s)\lambda_2^{i+k}(s)}{\lambda_2^{2k}(s) - \lambda_1^{2k}(s)}, \quad (35)$$

$$V_i(s) = \frac{\lambda_1^{i+k}(s) - \lambda_2^{i+k}(s)}{\lambda_1^{2k}(s) - \lambda_2^{2k}(s)}, \quad (36)$$

where $\lambda_{1,2}(s) = \left[1 \pm \sqrt{1-4a(1-a)s^2}\right]/(2as)$.

Considering that our Markov chain starts at state $i = 0$, $u_{0,n}$ and $v_{0,n}$ can be written as:

$$u_{0,n} = \frac{\mathrm{d}^n U_0(s)}{n!(\mathrm{d}s)^n}|_{s=0}, \, v_{0,n} = \frac{\mathrm{d}^n V_0(s)}{n!(\mathrm{d}s)^n}|_{s=0}, \quad (37)$$

which can be written in closed-form as in (6) and (7). □