The Cost of Free Spectrum

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Abstract

There has been growing interest in increasing the amount of wireless spectrum available for unlicensed access, especially lower frequency "prime" spectrum which could be used to offer wide area coverage similar to licensed cellular providers. While this additional unlicensed spectrum allows for market expansion, it also influences competition among providers and can increase congestion (interference) among consumers of wireless services. We study the value (social welfare and consumer surplus) obtained by adding unlicensed spectrum to an existing allocation of licensed spectrum among incumbent service providers. We assume populations of customers who choose a provider based on minimum *delivered price*, given by the weighted sum of the price of the service and a congestion cost, which depends on the number of subscribers in a band. We consider both models in which this weighting is homogeneous across the customer population and where the weighting is heterogeneous, reflecting customers preference for a high or low class of service. For the models considered, we find that the social welfare depends on the amount of additional unlicensed spectrum, and can actually decrease over a significant range of unlicensed bandwidths. Furthermore, in a heterogeneous model introducing unlicensed spectrum can also reduce consumer welfare.

1   Introduction

The increase in demand for mobile data, driven in part by the proliferation of smart phones and tablets, is straining the capabilities of current broadband wireless networks. Service providers have consequently requested increases in the amount of spectrum allocated to commercial broadband services. That has in turn motivated numerous discussions concerning policies that would facilitate more efficient use of spectrum PCAST (2012), Peha (2009),

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Hazlett and Leo (2010), Berry et al. (2010), Bykowsky et al. (2010). A key policy distinction is whether such new spectrum is *licensed* or *unlicensed.* Licensed spectrum provides the license holder with exclusive access and is used, for example, to provide cellular services. Unlicensed spectrum (also referred to as "open access" or "commons" spectrum) can be used by any device (e.g., for WiFi access) that abides by certain technical restrictions, such as a limit on transmit power.

The unlicensed bands currently used by WiFi devices are at relatively high frequencies (i.e., 2.4 GHz and 5 GHz) and operate with low power restrictions, which limits their range to relatively short distances compared to the wide-area coverage of cellular services. There has been recent interest in allocating additional unlicensed spectrum at lower frequencies, in particular, the unused channels, or "white spaces" that lie within spectrum allocated to broadcast television.[1] Because radio signals tend to propagate further at lower frequencies, the broadcast television bands are more suitable for wide-area coverage than the current WiFi bands. [2] Allocating these bands for unlicensed access would lower the barriers faced by new entrants seeking to provide wireless data services. This is in contrast to licensed spectrum for cellular service, which must be purchased by auction or by negotiations with another Service Provider (SP), posing a steep barrier to entry.[3]

Adding new entrants to the market increases competition, leading proponents for unlicensed spectrum to argue that it will benefit consumers as well as the overall economy (e.g., see Milgrom et al. (2011)). However, spectrum is a congestible resource in the sense that shared use generates externalities due to interference. Hence the Quality of Service (QoS) for a particular user (measured in terms of throughput and/or latency) generally degrades as the number of users sharing the spectrum increases.[4] The high demand for wide-area access to wireless data services combined with open access to lower frequency bands could create excessive congestion in those bands leading to a "tragedy of the commons". Indeed this is one of the main arguments for granting exclusive-use licenses for spectrum.

It is unclear *a priori* which of the preceding effects will dominate and how this depends on the amount of unlicensed spectrum and consumer demand. In this paper, we introduce a model to gain insight into such questions. More precisely, we consider a market in which

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[1]In 2010 the United States Federal Communications Commission (FCC) published final rules for unlicensed use of white spaces (FCC (2010)).

[2]For example, the IEEE 802.22 standard being developed for white spaces can support distances greater than 30 km, similar to commercial cellular (IEEE (2013)).

[3]For example, in 2008 firms paid more that 19 billion in an auction for 1090 licenses within the 700 Mhz band. The majority of those licenses were purchased by incumbent cellular providers FCC (2008).

[4]In addition, the network infrastructure, which must carry the wireless traffic, also has a capacity limit, potentially introducing another source of congestion. Here we are mainly concerned with the effect of interference on QoS.

incumbent SPs compete for a common pool of consumers. Each SP has an existing allocation of licensed spectrum, and we evaluate the effect of introducing unlicensed spectrum as an additional resource. Any incumbent SP as well as new entrants, may offer service in the unlicensed band in addition to its licensed band, modeling the fact that this band has a low barrier to entry. To capture congestion effects, we assume that consumers in a particular band experience a congestion cost that depends on the total number (or mass) of customers assigned to that band. All customers in a licensed band are served by the associated SP, whereas the customers in the unlicensed spectrum may be served by different SPs. Our goal is to determine how the additional unlicensed spectrum affects both social (total) and consumer welfare.

Our analysis builds upon the framework for price competition in markets for congestible resources developed in the operations, economics and transportation literature; see, for example Levhari and Luski (1978), Armony and Haviv (2003), Hayrapetyan et al. (2005), Acemoglu and Ozdaglar (2007a), Allon and Federgruen (2007), Xiao et al. (2007), Johari et al. (2010) and the discussion at the end of this section. In this framework, customers request service from firms (SPs) based on a *delivered price* that depends on the price paid for the service, announced by the SP, and the congestion cost. The firms then set prices to maximize revenue. The unlicensed spectrum can be viewed as an additional *non-exclusive* resource made available to each firm.[5] In contrast, prior work on congestible resources has generally assumed that each firm only has access to a resource for exclusive use.

To model customer choices, we assume that the delivered price of a service is a linear combination of the SPs announced price and the congestion cost. We consider the following two cases: (1) a *homogeneous* customer population in which all customers weight the congestion cost and announced price in the same way, i.e, all customers see the same delivered price; and (2) a *heterogeneous* customer population in which there are two user groups ("high-" and "low-QoS") with different price-congestion trade-offs. In the heterogeneous model, adding unlicensed spectrum could conceivably cause the market to segment, namely, by assigning users desiring higher (lower) QoS to licensed (unlicensed) spectrum. We will see that congestion causes the social welfare and consumer surplus to exhibit relatively complicated behavior.

Our main results are summarized as follows:

1. The social welfare depends on the amount of unlicensed spectrum that is added to the market. Adding an amount of unlicensed spectrum in a particular range, starting from zero, can cause the social welfare to *decrease*.

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[5]In this sense unlicensed spectrum is a *congestible public good*, e.g., see Scotchmer (1985).

2. In the homogeneous model, consumer surplus is a non-decreasing function of the amount of unlicensed spectrum.

3. In the heterogeneous model, *both* SP profit *and* consumer surplus can decrease.

4. In the heterogeneous model, the customer surplus can be a complicated, non-monotonic function of the amount of unlicensed spectrum added. (There can be many break points between which the customer surplus increases, decreases, or stays the same.)

The first result is perhaps counter-intuitive, and is reminiscent of *Braess's paradox* (Braess (1968)) in transportation networks: adding resources can decrease total system utility. A key difference here is that this decrease is caused by price setting by the SPs rather than by the users as in Braess (1968). The explanation for this decrease is that the incumbent SPs, when faced with new competition from the unlicensed band, may have an incentive to raise (instead of lower) their prices, depending on the amount of bandwidth. That facilitates a shift of traffic to the unlicensed band, where the associated interference externality is then shared with other SPs, and causes the overall welfare to decrease. The second result implies that with homogeneous customers, any such loss in total welfare consists solely of the loss in the SP's profits from serving fewer customers after raising its price.

In the homogeneous model, prices change continuously as a function of the amount of unlicensed spectrum being added. In the heterogeneous model, an SP may have an incentive to increase its price discontinuously in order to switch from serving both high- and low-QoS customers to serving a smaller number of high-QoS customers. This shifts more low-QoS customers to the unlicensed band increasing congestion there. Hence, when this switch happens, customer surplus decreases along with SP profits. Furthermore, the surplus can be strictly smaller than with no unlicensed spectrum. This is summarized by the preceding third and fourth results. Overall, these results suggest that adding new spectrum as a commons to existing allocations can affect social and customer welfare in complicated ways, and may have unexpected effects.

As alluded to previously, the general framework of competition with congestible resources has been studied in a number of different settings. In the context of service industries, work such as Levhari and Luski (1978) and Armony and Haviv (2003) considers models of competition where a firm's customers experience a latency given by a queueing delay. As in our work, customers select firms based on a linear combination of latency and price, where the weights in this combination can be heterogeneous across the customers.[6] Other related work considers models in which firms commit to a given latency and then incur a cost to

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[6]In Levhari and Luski (1978) each customer's weight is chosen from a distribution with continuous support, whereas in Armony and Haviv (2003) there are two customer classes as assumed here.

meet this commitment based on the number of customers they attract. (See Lederer and Li (1997) and Allon and Federgruen (2007) for a survey of this area). Here, we do not allow for such commitments.

Closer to our application is work motivated by communication and transportation networks. For example, Engel and Galetovic (1999) considers models in which privately owned toll roads compete for customers (drivers) who select roads based on the delivered price, whereas Acemoglu and Ozdaglar (2007a,b), Hayrapetyan et al. (2005) are motivated by communication networks such as the Internet, where different links may be owned by different SPs, and again customers select links based on a delivered price. A theme in much of that work is to characterize the inefficiencies that occur due to oligopolistic competition in congested markets, compared to the outcome under a benevolent social planner. In contrast, here we focus on the impact of adding unlicensed spectrum on the outcome of such oligopolistic competition. There have been similar studies of efficiency loss in so-called *selfish routing* models without pricing (e.g., Roughgarden and Tardos (2002)), and where prices are set by a benevolent manager (e.g., Cole et al. (2003)). That class of models has also been extended to allow for investment on the part of SPs as well as pricing decisions (e.g., Campo-Rembado and Sundararajan (2004), Johari et al. (2010), Acemoglu et al. (2009), Xiao et al. (2007). Here we assume that any investment is a sunk cost and focus solely on the pricing behavior of SPs.

Most of the aforementioned work related to communication networks is motivated by wire-line networks as opposed to the wireless setting we consider. An exception is Campo-Rembado and Sun (2004), which considers price and capacity allocation between two wireless SPs, each with licensed spectrum. Users respond to the sum of a price and a congestion term that reflects the probability that a user's service is blocked. Another related model can be found in Maillé and Tuffin (2010), which studies price competition between licensed wireless SPs. Users respond to "perceived prices," which depend on congestion as well as an announced price, but the relationship is not linear as in our model. Other work on competition among wireless SPs focuses on different issues such as the impact of auction design on competition Cramton et al. (2011), the effect of roaming agreements and termination charges Laffont and Tirole (2001), Armstrong and Wright (2009), and the impact of customer switching costs Shi et al. (2006).

The rest of the paper is organized as follows. Section 2 introduces our model, and Section 3 studies and compares social welfare and consumer surplus within this framework. Conclusions are given in Section 4, and some proofs and numerical calculations are provided in the appendix.

2   The Model

Here we present the heterogeneous model, where there are two classes of consumers with different sensitivities to delay. The homogeneous model is then presented as a special case.

Service Providers

We assume a set of incumbent SPs, each of which has its own licensed band, and a set of new entrants, which do not have licensed spectrum and must use the unlicensed band to offer service. (For tractability our analysis will primarily assume a single incument.) Each SP competes for customers by announcing a price for using its licensed band, and another price for using the unlicensed band. The SP then serves all customers who accept their posted price.[7] Suppose an SP i sets price p_i for service in its licensed band, price p_i^w in the unlicensed band, and serves x_i and x_i^w customers in those bands, respectively. Then i's revenue is given by $\pi_i = p_i x_i + p_i^w x_i^w$. (Here w stands for "white space".)

There is a congestion externality due to the interference suffered by customers in both the licensed and unlicensed bands. If SP i's licensed band serves a mass of customers x_i, then each customer served in this band experiences a *congestion cost* $l_i(x_i)$, which depends on the bandwidth and the technology deployed by SP i. The congestion cost in the unlicensed band, however, depends on the *total* mass of customers served in that band by *all* SPs. Specifically, letting x_i^w be the mass of customers served by SP i in the unlicensed band, the congestion cost for customers served in the unlicensed band is $g(X^w)$ where $X^w = \sum_{i \in \mathcal{N}} x_i^w$. The congestion cost $g(X^w)$ also depends on the bandwidth of the unlicensed band and the technology, which we assume is the same for all SPs. In this paper we consider the case where l_i is fixed and g varies according to the available bandwidth of the unlicensed spectrum (in a manner to be specified). Throughout the paper we assume that for a given bandwidth, all congestion costs are monotonically increasing and convex functions of the load (mass of customers served).

Customers

We consider a simple model for heterogeneous customers in which there are two different classes of customers, delay sensitive (high Quality of Service) and delay insensitive (low Quality of Service). Customers choose an SP based on the *delivered price*, which is the weighted sum of the price announced by an SP and the congestion cost she experiences when served by that SP.

Specifically, for a customer of type $t \in \{h, l\}$ (high, low) served by SP i, the *delivered price* in the licensed band is $p_i + \lambda_t l_i(x_i)$, where λ_t is the relative weight, and $\lambda_h > \lambda_l$. The delivered

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[7]The services offered by a particular SP in the licensed and unlicensed are treated as being distinct. Alternatively, the SP could announce one price for service and a rule for splitting its customers between the two bands.

price in the unlicensed band is $p_i^w + \lambda_t g(X^w)$. Customers within each class (high/low) choose the SP and type of service (licensed or unlicensed) with the lowest delivered price. When facing the same delivered price from multiple SPs, customers randomly choose one of the SPs.

The demand for services from the two classes is given by two downward sloping demand functions $D_h(p)$ and $D_l(p)$ with inverse functions $P_h(q)$ and $P_l(q)$, respectively.[8] A special case of this *heterogeneous* customer model is the *homogeneous* model in which $\lambda_h = \lambda_l$, so there is only one type of customer. (Alternatively, one of the demand functions can be set to zero.)

Pricing Game and Equilibrium

We consider a game in which SPs first simultaneously announce prices on licensed and unlicensed bands. Customers then choose SPs based on the delivered price. In this section we characterize the corresponding equilibrium along with the associated social welfare and consumer surplus.

Let x_i^h, x_i^l be the number (measure) of customers of each type that receive service from SP i in the licensed band. Similarly, let x_i^{wh}, x_i^{wl} be the number of customers of each type served by SP i in the unlicensed band. Thus, $x_i = x_i^h + x_i^l$ and $x_i^w = x_i^{wh} + x_i^{wl}$. We assume each customer is infinitesimally small and adopt the notion of Wardrop equilibrium to characterize how demand is allocated (Wardrop (1952)). Namely, given a price vector $(\mathbf{p}, \mathbf{p}^w)$, the non-negative demand vector $(\mathbf{x^h}, \mathbf{x^l}, \mathbf{x^{wh}}, \mathbf{x^{wl}})$ induced by $(\mathbf{p}, \mathbf{p}^w)$ must satisfy in the licensed bands:

$$
\begin{aligned}
p_i + \lambda_t l_i(x_i) = P_t(Q_t) \qquad & \text{if } x_i^t > 0, \ t \in \{h, l\} \\
p_i + \lambda_t l_i(x_i) \geq P_t(Q_t) \qquad & \text{if } x_i^t = 0, \ t \in \{h, l\}
\end{aligned}
\tag{1}
$$

and in the unlicensed bands:

$$
\begin{aligned}
p_i^w + \lambda_t g(X^w) = P_t(Q_t) \qquad & \text{if } x_i^t > 0, \ t \in \{h, l\} \\
p_i^w + \lambda_t g(X^w) \geq P_t(Q_t) \qquad & \text{if } x_i^t = 0, \ t \in \{h, l\},
\end{aligned}
\tag{2}
$$

where $Q_t = \sum_i (x_i^t + x_i^{wt})$ for $t \in \{h, l\}$ is the total number of customers of type t served in the market.

Remark: It is straightforward to show that given a price vector, the corresponding demand vector satisfying the above conditions always exists and is the solution to a convex program.[9]

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[8]In other words, $P_t(q)$ gives the maximum delivered price for which q customers of type t would be willing to purchase service.

[9]See, for example, Acemoglu and Ozdaglar (2007a) for a proof of a similar result without the addition of

We next define the equilibrium notion we will use for the overall game.

Definition 2.1 *A pair* $(\mathbf{p}, \mathbf{p^w})$ *and* $(\mathbf{x^h}, \mathbf{x^l}, \mathbf{x^{wh}}, \mathbf{x^{wl}})$ *is a pure strategy Nash equilibrium if* $(\mathbf{x^h}, \mathbf{x^l}, \mathbf{x^{wh}}, \mathbf{x^{wl}})$ *satisfies equation (1) and (2) given* $(\mathbf{p}, \mathbf{p^w})$, *and no SP can increase its revenue by changing its prices.*

Social Welfare and Customer Surplus

Next, we define the notions of *social welfare* and *customer surplus* in this setting.

Definition 2.2 *Let* $(\mathbf{x^h}, \mathbf{x^l}, \mathbf{x^{wh}}, \mathbf{x^{wl}})$ *be the demand vector induced by some price vector* $(\mathbf{p}, \mathbf{p^w})$ *according to (1) and (2). Then the* social welfare *is given by*

$$
\begin{aligned}
SW = &\int_0^{Q_h} P_h(q)dq + \int_0^{Q_l} P_l(q)dq \\
&- \sum_{i \in \mathcal{N}} \lambda_h x_i^h l_i(x_i) - \sum_{i \in \mathcal{N}} \lambda_l x_i^l l_i(x_i) - \lambda_h g(X^w)X^{wh} - \lambda_l g(X^w)X^{wl},
\end{aligned} \tag{3}
$$

where $X^w = X^{wh} + X^{wl}$, *and* X^{wh} *and* X^{wl} *are the number of high and low customers in the unlicensed band, respectively, and* $Q_h = \sum_{i \in \mathcal{N}} x_i^h + X^{wh}$ *and* $Q_l = \sum_{i \in \mathcal{N}} x_i^l + X^{wl}$ *are the total number of high and low customers served in the market.*

Definition 2.3 *Given price and demand vectors, let* D_h *and* D_l *be the resulting delivered price for high and low customers, respectively. Then the customer surplus is given by*

$$
CS = \int_0^{Q_h} (P_h(q) - D_h)dq + \int_0^{Q_l} (P_l(q) - D_l)dq \tag{4}
$$

where Q_h *and* Q_l *are defined in Definition 2.2.*

Figure 1 shows an example of the pricing game, where there are two incumbents with latency functions $l_1(\cdot)$ and $l_2(\cdot)$ in their respective licensed bands. The latency function of the white space is given by $g(\cdot)$. The customer population is homogeneous with the single inverse demand curve $P(q)$. At an equilibrium in which all bands are used, the Wardrop conditions imply that the delivered prices across the licensed and unlicensed bands are the same. That is,

$$
l_1(x_1) + p_1 = l_2(x_2) + p_2 = g(X^w) + p^W.
$$

Figure 1 shows that the price charged in the unlicensed band, $p^W = 0$. We will show in the next section that this is true at any Nash equilibrium. The revenue of SP i is $\pi_i = p_i x_i$,
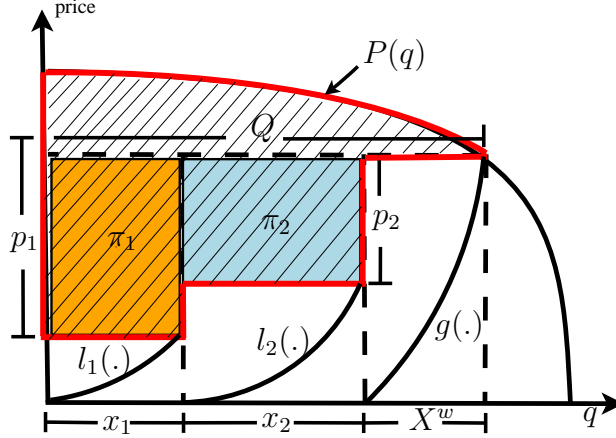
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an analogous unlicensed resource.

Figure 1: Illustration of pricing game with two SPs, homogeneous customers and unlicensed spectrum.

$i = 1, 2$, and corresponds to the area of the indicated rectangle. The social welfare is the shaded area shown in the figure. The consumer surplus is the shaded area above the horizontal line at the equilibrium delivered price, and is equal to the social welfare minus the revenue of all SPs.

3   Main results

We first show that the announced price in equilibrium in the unlicensed band is the marginal cost, which we assume to be zero. We then use this result to investigate the effect of adding unlicensed spectrum on the social welfare and consumer surplus.

3.1   Equilibrium Price in Unlicensed Spectrum

Let $\mathbf{p}^*$ and $(\mathbf{x^{h*}}, \mathbf{x^{l*}})$ denote the equilibrium price and demand vectors in the licensed bands, respectively, and $\mathbf{p}^{w*}$ and $(\mathbf{x^{wh*}}, \mathbf{x^{wl*}})$ denote the corresponding equilibrium prices and demands in the unlicensed band. Let Q_t^* be the total number of type t customers served in the equilibrium. Also recall that $g(\cdot)$ is the congestion cost in the unlicensed band; $g(0) > 0$ then represents some fixed cost experienced by all customers in that band.

Lemma 3.1 *Given at least two SPs in a market with unlicensed spectrum, at any equilibrium either $\mathbf{p}^{w*} = \mathbf{0}$, with at least two SPs serving a positive mass of customers, or no customers are served in that band. Furthermore, if no customers are served, then $g(0) \geq P_t(Q_t^*)$ for both types of customers.*

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Remark: This result can be easily extended to a scenario where each SP has a non-zero marginal cost for providing the unlicensed service. In that case, the equilibrium unlicensed price will be the marginal cost. The qualitative result is the same, and thus for simplicity, we assume the marginal costs are zero. (This can be interpreted as having zero-cost devices.)

The intuition behind this result is that because all customers in the unlicensed spectrum experience the same congestion cost, the SPs with the lowest price capture the entire market for a given customer class. Therefore, if there are more than two SPs serving customers in the unlicensed band, competition will drive the prices to zero. However, if all the SPs competing in the unlicensed band also offer services in licensed spectrum, then lowering the price for unlicensed service can influence their revenue in the licensed band. We next provide a formal argument that covers this case. The second possibility is that no one offers service in the unlicensed band; for this to be the case, it must be that no matter what price is charged, the delivered price in the unlicensed band ($g(0)$) exceeds that in the licensed bands ($P_t(Q_t^*)$). If the congestion cost in the unlicensed band satisfies $g(0) = 0$, then the second possibility cannot occur.

Proof of Lemma 3.1

Assume that in equilibrium $\mathbf{p}^{w*} \neq \mathbf{0}$ and some SP serves a strictly positive mass of customers. Call an SP *active* if in equilibrium she sets a positive price that results in a strictly positive quantity of customers. The Wardrop equilibrium conditions imply that all active SPs in the unlicensed band must charge the same price, $p^{w*} > 0$.

Furthermore, in equilibrium if one SP is active in the unlicensed band, then all SPs must be active in that band. Otherwise, an inactive SP could increase its revenue by charging the same price in the unlicensed band. That would increase the number of customers without decreasing revenue from the licensed band. Thus it follows that in any equilibrium either all SPs are active and charge the same price $p^{w*} > 0$, or no SP serves a positive mass of customers in the unlicensed band. The latter case contradicts our assumption, hence we focus on the first case.

Given that all SPs are active and charge the same price in the unlicensed band, consider the effect of an SP i dropping her price by ϵ. It must be that before dropping her price she is serving $x_i^w < X^w$. Since all customers experience the same latency, all X^w customers will switch to SP i. Furthermore, some customers currently on licensed bands will also switch to i's unlicensed service.

It follows that if SP i is an entrant, then she can significantly increase her revenue by dropping the price by a small amount. However, if SP i is an incumbent, dropping her price in the unlicensed band by ϵ can influence her revenue in the licensed spectrum. Nevertheless, if ϵ is small enough, we next show that SP i still improves its revenue. Let the equilibrium

price in the licensed band of SP i be p_i^*, and assume it keeps this price. Suppose that after dropping the price in the unlicensed band by ϵ the customer mass of SP i in the licensed band is reduced by Δ_{x_i}. The overall change in i's profit π_i is given by

$$\Delta_{\pi_i} \geq (p^{w*} - \epsilon)X^{w*} - (p^{w*}x_i^{w*} + \Delta_{x_i}p_i^*)$$
$$= p^{w*}(X^{w*} - x_i^{w*}) - \epsilon X^{w*} - \Delta_{x_i}p_i^*.$$

Since $\lim_{\epsilon \to 0} \Delta_{x_i} = 0$, there exists a sufficiently small $\epsilon > 0$ such that Δ_{π_i} is strictly positive. Thus, decreasing the price in the unlicensed band is a profitable deviation for SP i. This contradicts the initial assumption. Therefore, $p_i^{w*} = 0$ for every SP i serving customers in the unlicensed band. Furthermore, for this to be an equilibrium at least two SPs must serve customers at this price.[10]

Finally, to prove the last part of the lemma, suppose that no SP is serving customers in the unlicensed band in equilibrium, but $g(0) < P_t(Q_t^*)$. An incumbent SP could then increase its revenue by offering unlicensed service. Hence any incumbent SP that deviated to offer unlicensed service would be the sole provider of unlicensed service. Suppose that such a provider maximizes its revenue across the licensed and unlicensed bands while keeping the total number of customers served fixed (so as to not change the delivered price). Inspecting the optimality conditions of this problem, it can be seen that the incumbent can always increase its revenue by using both bands, leading to a contradiction. ∎

3.2 Social Welfare

Next we analyze the social welfare. Our main insight is that adding unlicensed spectrum can have nontrivial effects on the competition among SPs. As discussed in the introduction, our results show that adding unlicensed spectrum can decrease the total social welfare. Furthermore, we analyze how the social welfare varies depending on the amount of unlicensed spectrum added. We start with the simple case of homogeneous consumers and show that for a class of demand and latency functions, social welfare first decreases and then increases as more unlicensed spectrum is added. We then illustrate that in a model with heterogeneous customers the total welfare of the system as a function of additional unlicensed spectrum can vary in a more complicated manner.

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[10]SPs would be indifferent between announcing a price of zero or an arbitrarily high price for unlicensed service, since in either case their revenue is zero.

3.2.1 Homogeneous model

Suppose an incumbent (monopoly) operates in a licensed band before an unlicensed band becomes available. Once the unlicensed band is open, one or more entrants can then enter the market using this band only. The incumbent can also offer service in the unlicensed band. We will characterize the variation in the social welfare in such a setting with the restrictions on the consumer demand and latency functions to be described.

In this section we focus on the homogeneous model and so without loss of generality we set $\lambda_l = \lambda_h = 1$ and drop the customer type from the notation. As in Acemoglu and Ozdaglar (2007a), we consider the case where all users have the same valuation W for receiving service (i.e., users are homogeneous not only in how they weight congestion and price, but also in how they value the resulting service). This corresponds to the inverse demand $P(x)$ having constant value W for $0 \leq x \leq 1$ and then dropping to zero for $x > 1$, where without loss of generality we normalize the total mass of customers to be one. Customers choose an SP based on delivered price as long as it is at most W. We focus on the case where W is such that prior to adding the unlicensed band not all customers are served by the incumbent. This can occur because first, the incumbent is a monopolist and so has an incentive to limit supply to extract a higher price and, second, the congestion cost can be too high for all customers to be served.

For the congestion costs, we restrict our attention to linear costs, defined as follows. The incumbent operates in the licensed band with the congestion cost

$$l(x) = T_1 + bx,$$

where $b > 0$ and $0 \leq T_1 \leq W$. The bandwidth of the unlicensed band is $C \geq 0$. The congestion cost in the unlicensed band is

$$g(x) = T_2 + \alpha_C x,$$

where $0 \leq T_2 \leq W$. Here we assume that $\alpha_C > 0$ is decreasing in C; when no unlicensed spectrum is open then $\alpha_0 = \infty$, and as $C \to \infty$, $\alpha_C \to 0$. T_1, T_2 can be interpreted as the fixed costs of connecting to the SP. We also assume

$$g(1) > l(0) \text{ and } l(1) > g(0),$$

that is, the congestion cost of serving the whole market in one band exceeds the fixed cost of connecting in the other.

We will examine what happens when the incumbent's bandwidth, which can be translated

to the coefficient b of the congestion cost, is fixed and we vary C, the bandwidth allocated to the unlicensed band. We have the following result.

THEOREM 3.2 *Consider an incumbent SP with licensed spectrum that does not serve all of the demand. If an amount of unlicensed spectrum C is added then:*

(i) *For every $C \geq 0$ there is a unique equilibrium.*

(ii) *The social welfare at an equilibrium, $S(C)$, can be described as follows. There exist $0 < C_1 < C_2 \leq \infty$ such that $S(0) = S(C_1) > S(C_2)$ and $S(C) = S(0)$ for $0 \leq C \leq C_1$; $S(C)$ is monotone decreasing for $C_1 \leq C \leq C_2$; and $S(C)$ is monotone increasing for $C \geq C_2$.*
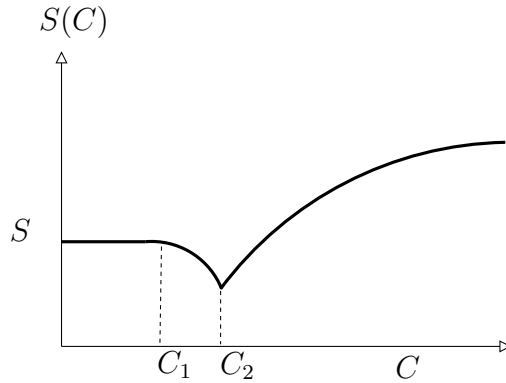

Figure 2: Social welfare as a function of the bandwidth of unlicensed spectrum.

Figure 2 illustrates the behavior of $S(C)$ as specified in this Theorem.

Remark: It can be shown that by introducing unlicensed spectrum, the efficiency can decrease to 62% of the social welfare with no unlicensed spectrum for the class of demand and latency functions assumed here. A precise analysis of this is given in Appendix B.1. The same qualitative result is also observed in more general classes of games that consist of multiple competing incumbents and a more general class of demand functions; see Appendix B.2 for a numerical example. Another possible generalization is to consider more general latency functions; keeping the remainder of assumptions in Theorem 3.2, it can be seen that a similar argument will go through if the latency of the incumbent is any differentiable, convex increasing function that satisfies $l(0) + \frac{1}{2}l'(0) > q(0)$, while the latency function for the unlicensed band if still linear. Allowing for a general latency function for the unlicensed band is more problematic as the revenue optimization problems faced by the incumbent may no longer be convex.

The formal proof of this theorem is provided in Appendix A.1. Below we provide an intuitive explanation.
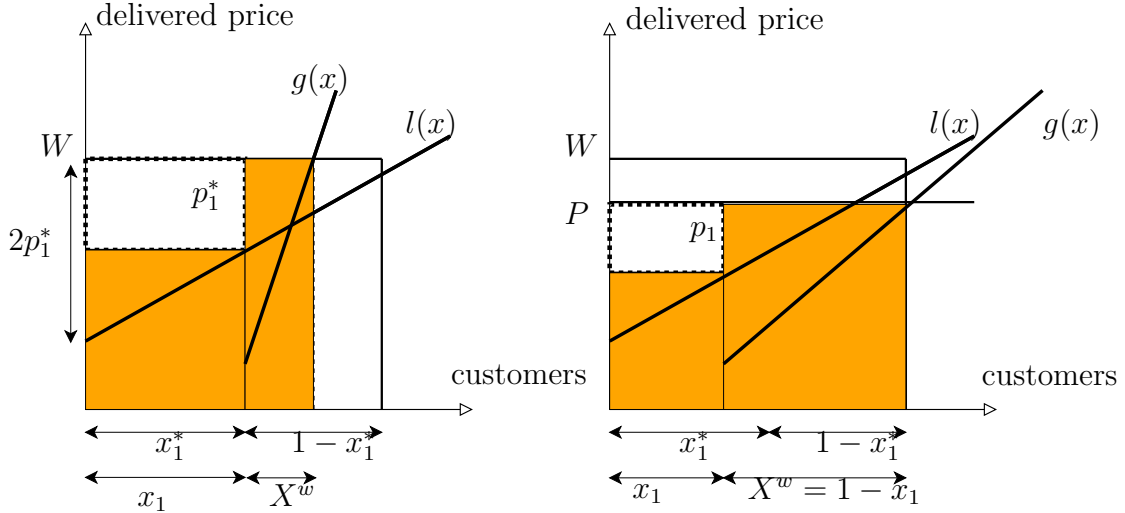
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Figure 3: Impact of adding an unlicensed band

Consider the two cases illustrated in Figure 3. One (left hand side) is where C is relatively small compared with the number of unserved customers. This results in a congestion cost with a steep slope. In this case the unlicensed band can only serve a fraction of the customers currently not served by the incumbent. Therefore, the unlicensed band does not create competition with the incumbent SP. The second case (right hand side) is where C is large. Here, service on the unlicensed band is good enough to attract the incumbent's customers. As a result, the incumbent loses some customers. Recall that in both cases, by Claim 3.1, the price in the unlicensed band is always zero.

There is an interesting transition between the two examples exhibited in Figure 3 as C increases. Let C_1 be the minimum value such that the congestion cost is low enough for service in the unlicensed band to be attractive to all customers not presently served by the incumbent. Observe that for all $0 \leq C \leq C_1$, the congestion cost in the unlicensed band is equal to W. Thus, even though the unlicensed band allows for a market expansion, because of high congestion cost, social surplus remains unchanged.

If we increase the bandwidth of the unlicensed band, it seems that congestion costs should decline and social welfare increase. Better service in the unlicensed band will attract the incumbent's customers. However, the incumbent need not respond to this erosion in share with a price cut. In fact, the incumbent might benefit from a price increase. This would drive even more customers into the unlicensed band, worsening the service quality there. Customers that remain in the licensed band pay more, but they receive a higher quality of service and have no incentive to use the unlicensed band. Because of this, the number of customers consuming lower quality service increases, which increases the overall congestion cost and reduces social welfare.

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An increase in bandwidth in the unlicensed band results in no increase in social welfare until C reaches a value C_2. Beyond this point, the quality of the unlicensed band is good enough so that if the incumbent keeps raising its price, it will lose too many customers. Because of the competition between the two types of services, the delivered price starts to decline. Falling delivered prices benefit customers and social welfare starts to increase.

3.2.2 Heterogeneous Model

In the homogeneous model, while social welfare is not a monotonic function of unlicensed spectrum bandwidth, it changes in a rather simple (unimodal) manner. We show that with heterogeneous customers, social welfare as a function of unlicensed band capacity is more complicated.

To illustrate we again consider a simple scenario, where a single incumbent (monopoly) operates on a licensed band before the unlicensed band is open. One or more entrants can then enter the market using the unlicensed band only. The incumbent can also offer services on the unlicensed band. As in Section 3.2.1, we again assume that all customers of the same type t have a common valuation W_t for service, but now we allow this valuation to vary across the two classes. Recall, this corresponds to the inverse demand $P_t(q)$ for each type t being a constant W_t for $0 \leq q \leq Q_t$ and then dropping to zero for $q \geq Q_t$, where Q_t is the total mass of type t customers, which we also allow to vary across the two classes. Customers with such a demand function choose an SP as long as its delivered price is at most W_t. Note that such a "box" demand function is uniquely determined by the tuple (W_t, Q_t). We will use this type of demand for two classes of customers ($t \in \{l, h\}$). Moreover, we assume $W_h > W_l$ and $Q_h < Q_l$, i.e., the high type customers have higher valuations for the service and there are more low type customers than high type customers.
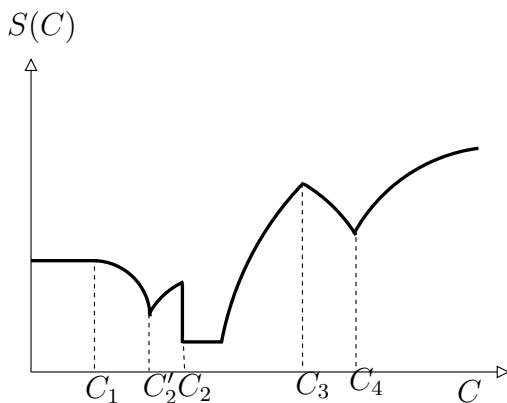

Figure 4: Social welfare as a function of unlicensed band's capacity for a heterogeneous model.

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Figure 4 illustrates a possible scenario, where the social welfare function changes in a complex manner as a function of the unlicensed band's capacity C. (A numerical example exhibiting this behavior is given in Appendix B.3.) Namely, when $0 \leq C \leq C_2$ the incumbent serves both types of consumers. The shape of the social welfare function is similar to that in the homogeneous model. When $C = C_2$, there is a sudden drop in the social welfare. At this threshold, the incumbent will increase the price discontinuously to exclusively serve the the high type customers. At this point, all low type customers switch to the unlicensed band. When C is increased from C_2 to C_3 congestion in unlicensed spectrum declines, which results in an increase in total social welfare. When $C > C_3$, the unlicensed spectrum starts to compete with the licensed band again and we get a similar effect as in the homogeneous model: social welfare starts to decrease until $C = C_4$, and then it starts to increase again.

The sudden increase in the price at $C = C_2$, and the drop of the social welfare will be analyzed more carefully in Section 3.3. Notice that adding more unlicensed spectrum increases the competition, and the revenue of the monopolist is always decreased. We will see that the drop of the social welfare at $C = C_2$ is caused by the decrease in *both* the monopolist's revenue and consumer surplus.

3.3   Consumer Surplus

We now turn to the question of how consumer surplus is affected by adding unlicensed spectrum. In this section we analyze the change in customer surplus when additional unlicensed spectrum is added to an existing market of wireless services offered in licensed spectrum. We show that when customers are homogeneous, adding additional unlicensed spectrum can never make the delivered price increase, which indicates that consumer surplus is non-decreasing. However, important difference emerges with the heterogeneous model: as the capacity of unlicensed spectrum changes, the incumbent SP may switch the types of customers she is serving to maximize her revenue. In particular, suppose that the incumbent SP serves both high and low type customers with no unlicensed band (to maximize her revenue), then intuitively, for an unlicensed band with small enough capacity, the incumbent will keep serving both classes of customers. While for an unlicensed band with large enough capacity, the incumbent will choose to raise the price and serve only high type customers, causing a drop in customer surplus. This effect is clearly absent in the homogeneous model and is a very robust prediction. In particular, we show that the effect described above is *always* present in heterogeneous models for general demand curves and latency functions.

More formally the results in this section are stated in Theorem 3.3 and Theorem 3.4. The first of these theorem follows and treats the homogeneous case.

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THEOREM 3.3 *Consider an incumbent SP with licensed spectrum in a homogeneous model, i.e. $\lambda_l = \lambda_h$. Let D_0, D_1 be the delivered prices at equilibrium before and after unlicensed spectrum is introduced, respectively, then $D_0 > D_1$.*

Proof: The theorem is proved by contradiction. First, we introduce the following notations. Let D be the delivered price for a homogeneous model with a single incumbent, where the demand and congestion can be general functions. Given such delivered price, let a be the number of customers in the unlicensed band and let x and p be the number of customer and the price of the licensed band, respectively, such that the resulting delivered price is D. (See Figure 5.)
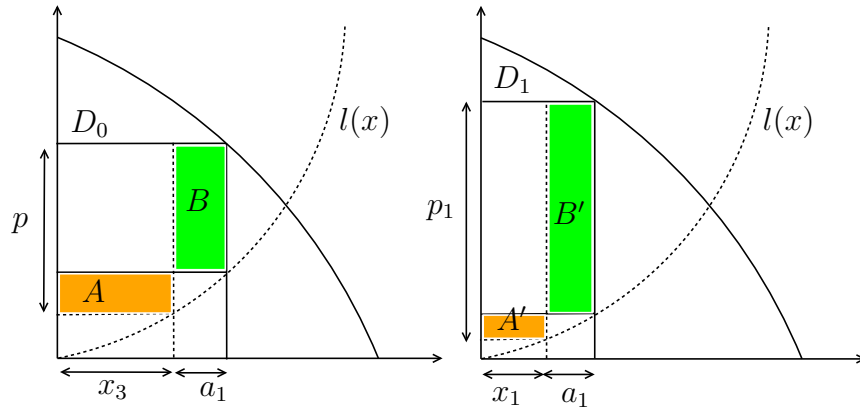

Figure 5: Illustrations of the revenue differences used in the proof of Theorem 3.3.

In particular, we will consider the following three pricing situations: (*i*) The first case is before the unlicensed spectrum is introduced and the monopoly maximizes its revenue. In this case, let D_0 be the delivered price. Because unlicensed spectrum was not introduced $a_0 = 0$ and $D_0 = l(x_0)+p_0$. (*ii*) The second case if after a fixed amount of unlicensed spectrum is introduced and the provider charges a price to maximize revenue under competition with the unlicensed spectrum. In this scenario, we denote the parameters by D_1, p_1, x_1, a_1. (*iii*) Lastly, we consider the situation where the provider charges a price p_2 so that under the competition with the unlicensed band, the delivered price is $D_2 = D_0$. In such a case, let x_2 and a_2 be the number of customers in the licensed and unlicensed band, respectively.

To establish a contradiction, we assume that $D_1 > D_0 = D_2$ and show that this implies $p_2 x_2 > p_1 x_1$. This is a contradiction because the provider is assumed to maximize its revenue at p_1.

To see this, we first observe that because $D_1 > D_2$, $g(a_1) = D_1$, and $g(a_2) = D_2$, it must be that $a_2 < a_1$, also since $D_1 > D_0$ the inverse demand curve at x_0 can not be flat and so it must be that $x_2 + a_2 = x_0$.

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Next, consider the function

$$R(x) = (D_2 - l(x))x = (D_0 - l(x))x,$$

which gives the revenue the incumbent can obtain when there is no unlicensed spectrum as a function x. Because $l(x)$ is a convex function, $R(x)$ is concave. Also, it is assumed that in the first scenario (before unlicensed spectrum is introduced) the provider achieves the maximum revenue of $R(x_0)$. Therefore, $R(x)$ is an increasing function from 0 to x_0. Thus, since $x_2 \leq x_0$, it must be that

$$R(x_2) > R(x_2 + a_2 - a_1) =: R(x_3).$$

Here we have defined $x_3 = x_2 + a_2 - a_1 < x_2$. Thus, $x_3 + a_1 = x_2 + a_2$. Now, because $D_2 = D_0 < D_1$, we have

$$x_2 + a_2 > x_1 + a_1.$$

Furthermore, because $x_3 + a_1 = x_2 + a_2 > x_1 + a_1$, we have $x_3 > x_1$.

Now consider the difference

$$R(x_3) - R(x_0) = R(x_3) - R(x_3 + a_1).$$

As seen on the left-hand side of Fig. 5, this is the difference between the area A and B, which is

$$R(x_3) - R(x_0) = x_3(l(x_3 + a_1) - l(x_3)) - a_1(D_0 - l(x_3 + a_1)). \tag{5}$$

Similarly, considering the difference $R(x_1) - R(x_1 + a_1)$ (given by the difference between the areas of A' and B' on the right side of Figure 5), we have

$$\begin{aligned} p_1 x_1 &- (D_1 - l(x_1 + a_1))(x_1 + a_1) \\ &= x_1(l(x_1 + a_1) - l(x_1)) - a_1(D_1 - l(x_1 + a_1)). \end{aligned} \tag{6}$$

Now since $x_3 > x_1$ and $l(x)$ is convex, we obtain

$$x_3(l(x_3 + a_1) - l(x_3)) > x_1(l(x_1 + a_1) - l(x_1)).$$

Furthermore, $D_0 < D_1$ and $x_3 + a_1 > x_1 + a_1$ implies

$$a_1(D_0 - l(x_3 + a_1)) < a_1(D_1 - l(x_1 + a_1)).$$

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Thus, the quantity in (6 must lower bound that in (5), giving

$$R(x_3) - R(x_0) > p_1 x_1 - (D_1 - l(x_1 + a_1))(x_1 + a_1).$$

Moreover, $(D_1 - l(x_1 + a_1))(x_1 + a_1)$ corresponds to the revenue the provider would achieve without the unlicensed band if it charges the price $D_1 - l(x_1 + a_1)$. Thus, $R(x_0) > (D_1 - l(x_1 + a_1))(x_1 + a_1)$. Therefore,

$$R(x_3) > p_1 x_1,$$

which is the desired contradiction to the fact that the provider optimizes its revenue.   ■

Our second theorem in this section describes a drop in customer surplus in the case of heterogeneous users.

THEOREM 3.4 *Consider a heterogeneous model with a single incumbent such that before unlicensed spectrum is introduced a mixture of different customer types is served at equilibrium. We also assume that $P_h(0) \geq \lambda_h l(0)$. Under these assumptions, there always exists a $C_0 > 0$ such that when the bandwidth of unlicensed is increased from C_0^- to C_0^+, the incumbent will increase her price discontinuously to serve high type customers exclusively which causes a drop in customer surplus.*

Proof: See Appendix A.2.

Notice that adding more unlicensed spectrum increases the competition, and the revenue of the monopolist is always decreased. Thus, this result shows that introducing additional spectrum as unlicensed band leads to a decrease in *both* the monopolist's revenue and consumer surplus.

In Appendix B.3 we provide some numerical examples to illustrate how the consumer welfare depends on the unlicensed spectrum's bandwidth. We show that there exists a range of parameters such that when C_0^+ bandwidth of unlicensed spectrum is introduced customer surplus is strictly smaller than when there was only licensed spectrum. Below, we explain the intuition behind the result of Theorem 3.4.

Consider the case prior to unlicensed spectrum being introduced. The provider charged a price p that serves both types of customers. In particular all high type customers and a fraction of the low type customers are served at this price. Let R be the total revenue in this scenario. Note that the provider can also charge a high enough price p_H such that at this price only high type customers use the service. Let the resulting revenue under p_H be R_H, where $R_H < R$, since by assumption the provider is serving both types of customers.

Now, some amount C of unlicensed spectrum is introduced, which creates competition with the provider. As a result the optimal revenue R_C will decrease in C. Observe that

when C is small the provider will also change the price p by a small amount such that at this price both type of users are still served, but when C is large enough the provider will have an incentive to suddenly increase the price and in many cases she could raise p up to p_H to eliminate low type customers and still obtain a revenue of R_H from the high types. When such a price increase occurs, high type customers need to pay a higher price thus their surplus is declines. On the other hand, the low type needs to use the unlicensed band, which will be highly congested. As a result, their surplus decreases as well.

To show the general result in Theorem 3.4 we need to prove that there is always a C_0 where the SP switches the class of customers she is targetting and when she does so that customer surplus drops. We provide the formal proof of this in Appendix A.2.

4   Conclusions

We have studied a model for the adding unlicensed spectrum to a market for wireless services in which incumbents have licensed spectrum. Our results highlight that the combination of free-entry in an unlicensed band and the congestible nature of this resource can lead to non-trivial behavior in terms of both overall welfare and consumer surplus. Namely, a type of Braess's paradox may occur in which social welfare decreases after the addition of this new resource; with heterogeneous customers, consumer surplus may also decrease. These results can be contrasted to markets without congestion, in which case adding supply will only improve welfare and the segmentation that emerges as a result of competition usually indicates improvement in efficiency or customer surplus.

From a policy point-of-view our analysis suggests that it should not be taken for granted that adding more unlicensed spectrum will lead to improvements in welfare. In some settings, such as rural areas, demand is naturally limited and so the type of congestion effect we consider may not be a significant issue. On the hand in areas where demand is high, perhaps alternative policies to limit congestion should be considered, such as establishing a market for a limited number of device permits (Peha (2009)). These conclusions apply only to one particular use case for unlicensed spectrum, namely offering service that competes with licensed providers. Of course, unlicensed spectrum could be used for other purposes that do not directly compete with licensed wide-area coverage. Indeed Milgrom et al. (2011) argue that making unlicensed spectrum available can help to foster innovation in technology and business models that may lead to unforeseen uses of this spectrum. Another factor that we have not modeled here is the investment decisions of providers. In particular different investment levels (or investments in different technology) could allow providers to differentiate their services in both the licensed and unlicensed bands.

A Missing Proofs

A.1 Proof of Theorem 3.2

In this proof we consider the incumbent as SP 1. Before the unlicensed band is introduced, let p_1^* be the price charged by the incumbent and $x_1^* < 1$ the mass of customers served. After opening the unlicensed band with bandwidth C, let x_1 and X^w be the number of customers using the licensed and unlicensed band respectively. Let p_1 be the price charged in the licensed band and P be the new delivered price. (See Figure 3).

Let C_1 be the value such that the corresponding congestion cost for $1 - x_1^*$ customers is equal to W. That is

$$g(1 - x_1^*) = T_2 + \alpha_{C_1}(1 - x_1^*) = W. \tag{7}$$

Proof of (i)

When $C \leq C_1$ the unlicensed band will not affect the price charged by the incumbent. Thus when $C \leq C_1$ the equilibrium is $p = p_1^*, x_1 = x_1^*, X^w = g^{-1}(W)$.

Next we establish uniqueness of the equilibrium and its structure for $C > C_1$. First we prove that when $C > C_1$, at any equilibrium, all the customers will be served. To see this, assume that $x_1 + X^w < 1$. We then know that the delivered price must be W, thus

$$g(X^w) = T_2 + \alpha_C X^w = W.$$

Because $C > C_1$ we have $X^w > 1 - x_1^*$. This shows that $x_1 < 1 - X^w < x_1^*$. Therefore the price

$$p_1 = W - l(x_1) > W - l(x_1^*) = p_1^*.$$

The incumbent, however, can charge a lower price to attract customers, who are currently unserved. Moreover, total revenue is a concave function [11], and it is maximized at p_1^*. Thus by lowering p_1, which is greater than p_1^*, the incumbent can gain more revenue. This leads to a contradiction.

We now show that there is a unique equilibrium. Assuming $C > C_1$, it follows that at

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[11]One can visualize the revenue of the incumbent $p_1^* x_1^*$ as the area of the dashed rectangle on the left picture of Figure 3, where its lower-right corner runs on the line $l(x)$. It is straightforward to see that the revenue function is a concave function.

any equilibrium (p_1, x_1, X^w) must satisfy:

$$x_1 + X^w = 1$$
$$l(x_1) + p_1 = T_1 + bx_1 + p_1 = P \leq W \tag{8}$$
$$g(X^w) = T_2 + \alpha_C X^w = P \leq W.$$

From this one can derive a revenue maximization problem for the incumbent. Given p_1, $x_1(p_1)$ satisfying the above equations is a linear function of p_1, thus $\pi_1(p_1) = p_1 x_1(p_1)$ is a quadratic function of p_1 and therefore the incumbent's problem is

$$\max_{p_1} \pi_1(p_1) \text{ subject to } p_1 + T_1 + bx_1(p_1) \leq W. \tag{9}$$

This problem always has an unique solution which yields uniqueness of the equilibrium.

Proof of (ii)

In the remainder of the proof we derive the behavior of $S(C)$. Observe that in optimization problem (9), depending on the parameters T_1, T_2, b, W, C, the solution must either be an interior point so that $\pi_1'(p_1) = 0$ or a boundary point satisfying $p_1 + T_1 + bx_1(p_1) = W$.

Now consider the solution of the unconstrained problem $\pi_1'(p_1) = 0$. From (8), we have

$$(b + \alpha_C)x_1 + p_1 = (T_2 - T_1) + \alpha_C.$$

Thus $\pi_1'(p_1) = (p_1 \cdot x_1(p_1))' = 0$ gives

$$p_1(C) = \frac{(T_2 - T_1) + \alpha_C}{2}; x_1(C) = \frac{(T_2 - T_1) + \alpha_C}{2(b + \alpha_C)}. \tag{10}$$

Because $l(1) > g(0)$, we have $T_2 - T_1 < b$, which shows that

$$x_1 = \frac{(T_2 - T_1) + \alpha_C}{2(b + \alpha_C)}$$

is increasing in α_C. Therefore, $p_1 + l(x_1)$ is increasing in α_C. However, α_C is a decreasing function of C, thus $p_1(C) + l(x_1(C))$ is decreasing in C. Furthermore, it is straightforward to see that when $C \to \infty$,

$$p_1(\infty) + l(x_1(\infty)) = \frac{(T_2 - T_1)}{2} + T_1 + b\frac{(T_2 - T_1)}{2b} = T_2 < W,$$

and when $C \to 0$ both $p_1(C)$ and $l(x_1(C))$ tend to infinity because $\alpha_0 = \infty$. Therefore there

exists an unique C^* such that $p_1(C^*) + l(x_1(C^*)) = W$.

Now, if $C^* \leq C_1$, then we define $C_2 = \infty$, otherwise we define $C_2 = C^*$. In both cases because $p_1(C) + l(x_1(C))$ decreases in C, we have for all $C \in [C_1, C_2]$

$$p_1(C) + l(x_1(C)) > p_1(C_2) + l(x_1(C_2)) = W.$$

Therefore the unique equilibrium determined by (9) needs to satisfy the condition that the delivered price is W.

Now, when the delivered price is W, observe that

$$g(X^w) = W \Rightarrow X^w = C(W - T_2).$$

Thus X^w increases in C and $x_1 = 1 - X^w$ decreases in C and by the same amount as X^w increases. However, $l(x_1) < W$, which means that when C increases the total mass of customers does not increase but some customers switch from a service with congestion cost $l(x_1)$ to a worse one (congestion cost of W) and thus the congestion cost increases and social welfare decreases.

Last, we consider the case $C > C_2$. We know that when $C > C_2$, the unique Nash equilibrium will satisfy $\pi_1'(p_1) = 0$ and we can use (10). In this case we know that the delivered price $P = p_1 + l(x_1) < W$, and all customers are served. Therefore, social welfare is

$$S(C) = p_1 x_1 + (W - p_1 - l(x_1)), \text{ here } l(x_1) = T_1 + bx_1.$$

One can take the derivative of $S(C)$ with respect to C. Here, we simplify the formulation by a change of variables. Namely, let $z = b + \alpha_C$ and $a = b + T_1 - T_2 > 0$. We have $z'(C) = \alpha'(C) < 0$ and

$$p_1(C) = \frac{z - a}{2}; x_1(C) = \frac{z - a}{2z}.$$

A simple calculation yields

$$S'(C) = z'(C)S'(z) = -\alpha'(C)\left(\frac{1}{4} + \frac{ab}{2z^2} + \frac{a^2}{4z^2}\right).$$

From this we see that $S'(C) > 0$, therefore $S(C)$ is an increasing function. This concludes the proof. ∎

A.2   Proof of Theorem 3.4

We first show that when the unlicensed spectrum bandwidth is large enough, the optimal revenue is obtained when the SP only serves high-type customers. Thus, because of the assumption that the SP serves a mixture of the two types in equilibrium before the unlicensed spectrum is introduced, there must be a bandwidth of unlicensed spectrum, C_0, at which the SP switches from serving both types to only the high type.

Given $\delta > 0$ let x_δ be the optimal point of

$$R_\delta = \max_x \{x \cdot (\delta - l(x))\}. \tag{11}$$
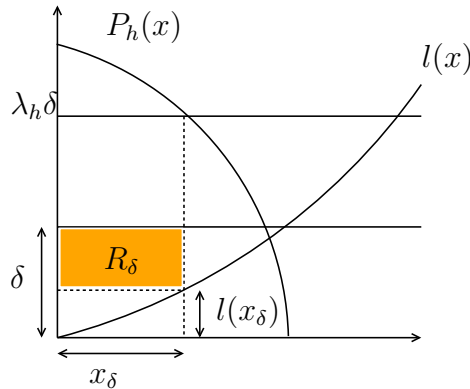
R_δ is the shaded area in Figure 6.

Figure 6: Improving revenue by targeting only high typed customers

Note that x_δ is unique because $l(x)$ is convex. Furthermore, x_δ is a continuous function of δ. When $\delta = l(0)$, $x_\delta = 0$ and there exists δ large enough such that $P_h(x_\delta) < \lambda_h \delta$. Note that because $P_h(x)$ is decreasing, $l(x)$ is increasing and $\lambda_h l(0) < P_h(0)$, one can see that there exists δ^* such that

$$P_h(x_{\delta^*}) = \lambda_h \delta^*.$$

Now consider a situation where there is only unlicensed spectrum and there are only low type customers. Let C_{δ^*} be a value such that if the bandwidth of unlicensed spectrum is C_{δ^*}, then the the congestion cost of the unlicensed band is δ^*. We will show that in the setting with the incumbent SP when $C = C_{\delta^*}$ the optimal revenue of the incumbent is obtained by serving high type customers only.

To see this, observe that when serving both types of customers, the delivered price for low type customers cannot be higher than $\lambda_l \delta^*$. This is true because we know that when serving both types of customers, the users in the unlicensed spectrum are of the low type and because we have C_{δ^*} bandwidth of unlicensed spectrum, the congestion in the unlicensed

band is at most δ^*. Therefore, the optimal revenue that the incumbent SP can obtain while serving both types can be at most

$$\max_x \{x \cdot \lambda_l(\delta^* - l(x))\} = \lambda_l R_{\delta^*},$$

where R_{δ^*} is defined in (11).

However, if the SP charges the price $p = \lambda_h(\delta^* - l(x_{\delta^*}))$, then the equilibrium of the game is the following: no low type customers use the licensed spectrum and x_{δ^*} high customers use the service in the licensed band. This is true because the congestion of the unlicensed band is $g(x) = \delta^*$, the delivered price for low type customers in the licensed band is

$$\lambda_l l(x_{\delta^*}) + \lambda_h(\delta^* - l(x_{\delta^*})) > \lambda_l l(x_{\delta^*}) + \lambda_l(\delta^* - l(x_{\delta^*})) = \lambda_l \delta^*.$$

Thus, no low type customers would choose to use the licensed band. On the other hand the delivered price for high type customers is $\lambda_l \delta^* = P_h(x_{\delta^*})$. Therefore no high type customers would use the unlicensed band either.

Now, in this case, the SP's revenue is

$$p \cdot x_{\delta^*} = \lambda_h R_{\delta^*} > \lambda_l R_{\delta^*}.$$

This shows that when $C = C_{\delta^*}$ the incumbent SP only serves high type customers. Therefore, there exists a $0 < C_0 < C_{\delta^*}$ such that if the unlicensed bandwidth is increased from C_0^- to C_0^+, the incumbent has an incentive to switch the class of customers and target only the high class.

Consider such a transition. When $C = C_0^+$ the unlicensed band is open to all low type customers and they do not have other choices. Thus the delivered price for low type customers must be non-decreasing compared with when $C = C_0^-$. Let x_h, x_l be the number of customers of high and low types in the licensed band and X^w be the number of customer in the unlicensed band at $C = C_0^-$. We have

$$\lambda_l l(x_h + x_l) + p = \lambda_l g(W^w).$$

Thus, the delivered price for the high type customers at that time is

$$\lambda_h l(x_h + x_l) + p < \frac{\lambda_h}{\lambda_l}(\lambda_l l(x_h + x_l) + p) = \lambda_h g(W^w).$$

This means that high type customers strictly prefer the licensed band to the unlicensed one.

Now, at $C = C_0^+$ the quality of the unlicensed band has worsened. Therefore, the

incumbent also has an incentive to raise the delivered price for high type customers. This shows that the delivered price for low type customers is non-decreasing and the delivered price for high type customers increases discontinuously. This concludes the proof. ■

B   Numerical Examples

B.1   An example of social welfare loss

Consider the case where $T_1 = T_2 = 0$, that is $l(x) = x, g(x) = \frac{x}{C}$. That is $\alpha_C = \frac{1}{C}$. We will calculate $C_1, C_2, S(0) = S(C_1)$ and $S(C_2)$ as functions of W.

First we know that at the optimal monopoly price p_1^*, we have

$$W - l(0) = 2p_1^* \text{ and } W = x_1^* + p_1^*$$

Thus $x_1^* = p_1^* = W/2$, and according to (7), we have

$$C_1 = \frac{1 - W/2}{W}.$$

Note that because we assume that before unlicensed spectrum is introduced, the incumbent did not serve all customer, this can only happen when $W < 2$. Now,

$$S(0) = S(C_1) = \frac{W^2}{4}.$$

Next to calculate C_2 , we have

$$p_1(C_2) + l(x_1(C_2)) = \frac{1}{2C_2} + \frac{1}{2(C_2 + 1)} = W,$$

which implies

$$C_2 = \frac{\sqrt{W^2 + 1} + 1 - W}{2W} > C_1.$$

Thus,

$$S(C_2) = \frac{W^2}{2(\sqrt{W^2 + 1} + 1)}.$$

For example if we consider $W = 1$, then before unlicensed spectrum is introduced, only half of the demand is met by a licensed spectrum with bandwidth 1. Adding $C_2 = \sqrt{2}/2 \sim 0.7$ capacity of unlicensed spectrum will create a new service that can serve all the demand. However, because of the congestion cost, the efficiency goes down to $\frac{S(C_2)}{S(C_1)} \sim 82.8\%$.

The worst example is when $W = 2$ then if $C_2 = \frac{\sqrt{5}-1}{4}$ bandwidth of unlicensed spectrum is open then the efficiency can go down to $\frac{S(C_2)}{S(C_1)} \sim 62\%$.

B.2 A numerical example with multiple, symmetric incumbents

We now give an example to show that the conclusions from Theorem 3.2 apply in more general settings. Specifically, we consider a scenario in which there is more than one incumbent SP. Additionally, we consider a linear inverse demand given by $P(q) = 1 - \beta q$, where β represents the elasticity of demand. Each SP has the same congestion cost in her licensed spectrum, with $l_i(x) = l(x) = x$ for all $i \in \mathcal{N}$. The congestion cost in the unlicensed band is given by $g(x) = \frac{x}{C}$.

For such a model there is a unique symmetric equilibrium. One can actually explicitly write down the social welfare with there are N SPs either with or without an additional band of unlicensed spectrum. Here, we provide a numerical example showing the social welfare decreases when additional capacity is added, i.e., Braess's paradox occurs. This example is shown in Fig. 7, where the solid curve is the welfare with additional unlicensed spectrum as a function of the amount of additional spectrum.
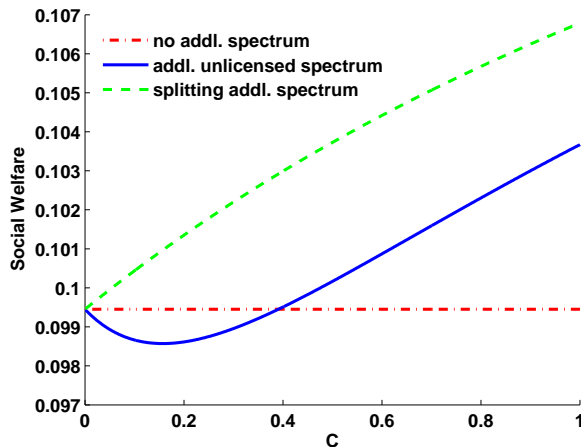

Figure 7: The social welfare in different scenarios as a function of additional capacity C in a symmetric linear network with $N = 2$ and $\beta = 4$.

We can also determine the social welfare for a scenario where instead of making the C units of capacity freely available, we divide this capacity evenly among the existing N SPs. We model this by again assuming that $l(x)$ is given by the customer mass per unit capacity for each licensed band, where initially the capacity is normalized to one. Hence, after giving each SP C/N additional units of capacity, the new congestion function is $\tilde{l}(x) = \frac{1}{1+C/N}x$. This quantity is also shown in Fig. 7. In this case dividing up the spectrum in this manner

improves the welfare for all values of C. This suggests that in cases where Braess's paradox occurs, licensing the spectrum to existing SPs can be socially more efficient.

B.3   An numerical for the heterogeneous model

We now show a numerical example illustrating the social welfare in a heterogeneous model. The specific numerical values are the following. Let $W_h = 1.6$, $Q_h = 1$, $W_l = 0.85$ and $Q_l = 1.3$, $\lambda_h = 0.4$ and $\lambda_l = 0.1$. Set $l(x) = x$ and $g(x) = x/C$. Without unlicensed band, it can be shown that the incumbent SP would set price $p_0 = 0.62$ to serve all of low class customers to maximize her revenue. The numerical results of social welfare, customer surplus and incumbent's price and revenue are plotted against the unlicensed band capacity C in Fig. 8.
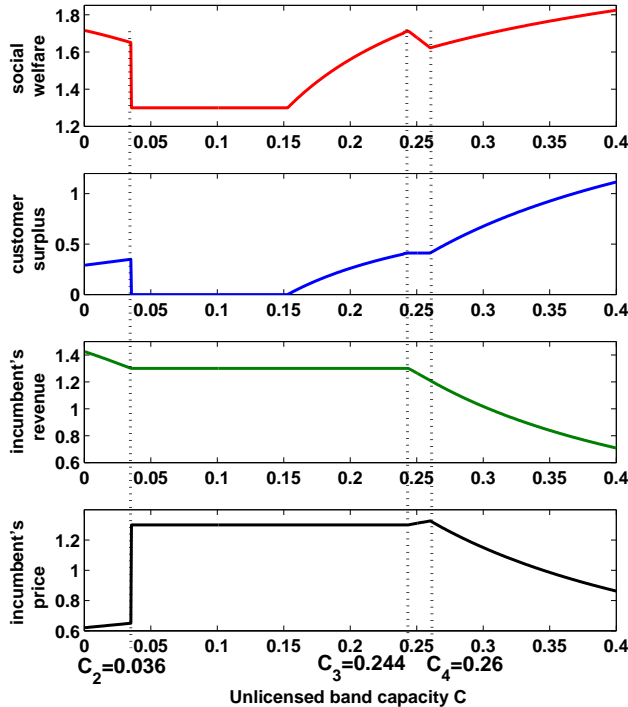

Figure 8: An example where the incumbent served both classes initially

In this numerical example, we found the social welfare, as a function of the capacity C in the unlicensed band, is not monotone. As shown in Fig. 8, there are two regions of C, $[0, C_2]$ and $[C_3, C_4]$ where social welfare decreases. Note that $C_2' > C_2$ for the parameters in this example, so we define $C_2' := C_2$.

Comparing social welfare and incumbent's price in Fig. 8, we find that the two regions where social welfare decreases are also where the incumbent's price rises. This phenomenon

is reminiscent to that in the model with homogeneous customers and can be explained in a similar way. Namely, the incumbent may benefit from raising her price since this may reduce the congestion in her licensed band and worsen the quality of service in the unlicensed band.

In particular, there are three stages as C increases.

Facing the competition from the service in the unlicensed band as C increases, the incumbent will eventually "retreat" from serving low class and suddenly increases its price to serve high class customers only to gain higher revenue. This corresponds to the jump in the monopoly's price and the drop in social welfare and customer surplus in Fig. 8 at C_2.

Thus the first stage corresponds to $C \in [0, C_2]$. In this stage, the service in the unlicensed band and that of the incumbent's licensed band will be competing on *low* class customers while the incumbent still serves all of the high class customers. Here we have the same observation that social welfare decreases as a result of the rise of the monopoly's price and congestion in the unlicensed band.

The second stage is when $C \in [C_2, C_3]$. This is the stage in which the market is sorted. Namely, the unlicensed band serves only low class and licensed band serves high class customers. Thus increasing capacity C has no impact on high class customers but improving the congestion in the unlicensed band. Therefore, the social welfare is constant or increasing in this stage.

Finally, when $C \in [C_3, \infty]$, the unlicensed band and licensed band will be competing on the *high* class customers while all of the low customers are being served in the unlicensed band. Similarly, the observation is that the social welfare decreases first as the increase of the monopoly's price until C reaches C_4 and then eventually starts to increase as the quality of service in the unlicensed band improves.

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