Complexity of Allocation Problems in Spectrum Markets with Interference Complementarities

Hang Zhou, Randall Berry, Michael L. Honig, and Rakesh Vohra

***Abstract*—Markets are often viewed as a key ingredient in facilitating more efficient dynamic spectrum access. In this paper we consider how such spectrum markets are influenced by a key property of the wireless medium: interference. Interference can result in ”complementarities” among the “spectrum goods” being traded, which complicates the design of an efficient market mechanism. We consider several alternative models for defining such spectrum goods, and explore the impact of these choices on the complexity of the resulting market.**

***Index Terms*—Dynamic Spectrum Sharing, Spectrum Markets, Optimization, Complexity**

I. Introduction

Spectrum markets have been proposed as a way to enable a more flexible allocation of spectrum [1]–[5], [25]. Such markets could be operated by a primary spectrum holder to lease spectrum for secondary use, or by a neutral third party that pools and leases spectrum from multiple providers and/or the government. Indeed, provisions for limited forms of such markets have been adopted in the U.S. [6]. When combined with software defined (frequency-agile) radio technology, such markets could be operated on much finer scales in time and space than traditional spectrum allocations. The design of such spectrum markets must account for the fact that transmitting in the same spectrum at nearby locations creates interference, which differentiates spectrum from many other goods. In particular, an agent's value for spectrum at a particular location may depend on the use of the spectrum at nearby locations. One solution to this is for a market to allocate a band of spectrum at only two locations that are sufficiently far apart, creating ”spatial guard zones” to mitigate interference. However, if spectrum is allocated on a small geographic scale, the overhead from such guard zones can become significant.[1] Moreover, the acceptable interference (and thus the appropriate guard zone) may vary greatly depending on the application and the technology used.

Instead of using predetermined guard zones to design markets without interference, we consider using markets to manage interference. For example, agents could purchase spectrum to preclude others from using it, creating guard zones on demand. This results in bundles of “spectrum assets” having *complementarities* (i.e., the value of a bundle may be greater than the sum of the values of the individual assets). In addition to creating guard zones, an agent could mitigate interference by coordinating transmissions across neighboring spectrum it owns, again leading to complementarities.[2]

Our focus in this paper is on developing models of different approaches to defining complementary spectrum assets in such a market. We study the effect of such definitions on the complexity of the resulting efficient allocation problems, where an efficient allocation is one that maximizes the value of the buyers in the market.[3] We are interested in the complexity of efficient market allocation because it is closely related to many mechanism design issues. In particular, if there exists a simple algorithm to solve the efficient allocation problem, then there may exist a simple pricing scheme (such as uniform pricing) that achieves the efficient allocation. Otherwise, it suggests that simple price functions cannot achieve the efficient allocation, which may require pricing *bundles* of goods in the market. Here, different definitions of spectrum assets result in different manifestations of interference complementarities. By comparing the complexities of the allocation problems in different markets, we examine the influence of the definitions of spectrum assets. In cases where the resulting allocation problems are NP-hard, we also discuss various algorithms for approximating the optimal algorithm. Our motive here is two-fold. First, such algorithms can be used to study the efficient allocation for a large number of assets and agents, and second, such algorithms might be useful in developing truthful mechanisms that approximate an efficient allocation [9], [10], [16]. Alternatively, the cases where the allocation problems are NP-hard may provide a reason to design spectrum markets in such a way that the problem sizes are not too large, so that despite the NP-hardness, the relevant problems can be solved exactly.

An alternative approach for dealing with such interference complementarities is via *bargaining*, i.e., allowing the agents to negotiate with each other to determine the how spec-

This research was supported in part by NSF under grants CNS-0519935 and CNS-0905407.

Hang Zhou, Randall Berry and Michael Honig are with Department of Electrical Engineering and Computer Science at Northwestern University (e-mails:{hang.zhou, rberry, mh}@eecs.northwestern.edu). Rakesh Vohra is with CMS-EMS, Kellogg School of Management at Northwestern University (e-mail:r-vohra@northwestern.edu).

Preliminary versions of this work were presented in part at the 2009 Annual Allerton Conference and 2011 IEEE WiOpt Conference.

[1] When spectrum is allocated on a large (e.g. national) scale as in traditional auctions, this is not a significant issue as such guard zones will be a small fraction of the total area allocated and boundaries can be drawn in sparsely populated areas.

[2] We focus on complementarities due to interference but note that complementarities between adjacent spatial locations can exist for other reasons as well, e.g., in traditional spectrum auctions complementarities exist due to the desire to create a national footprint in a given band. With more flexible radios and markets, this may be less of a concern.

[3] Of course, one might also be interested in maximizing the revenue of the seller. We do not explicitly address this here, but note that many mechanisms that attempt to maximize revenue are again closely related to solving the underlying efficient allocation problem.

trum assets are defined as well as how they are allocated. Such an approach dates back to the work of the economist Ronald Coase [7], who argued that if agents have well-defined property rights and there are no transaction costs, then such bargaining would lead to the efficient allocation. This may seem to suggest that the problem of defining spectrum assets is unimportant; would agents not simply bargain with each other to determine the correct definition? The problem with this conclusion is that transaction costs are not zero. In practice there are important *frictions* that can can impede bargaining such as the time needed to find counter parties and reach an agreement. When spectrum is allocated on a finer temporal or spatial scale, such frictions would likely increase. Indeed, in cases where the allocation problems are NP-hard, bargaining to achieve an efficient outcome is also likely to be difficult. Additionally, Coase's conclusions are based on agents having perfect information about each other's valuations and do not hold in general in the presence of imperfect information.

We first give a basic model in Section II for spectrum markets with complementarities, in which determining the efficient allocation is shown to be NP hard. Several approximation schemes are discussed. We then consider different models for defining the assets in a spectrum market, including allowing guard zones with secondary users (Section III) and allowing the market to determine a "radius" over which nodes may transmit (Section IV). We show that redefining the spectrum asset in this way leads to a substantial reduction in (worst-case) complexity.

In terms of related work, a number of papers have discussed mechanisms for allocating spectrum to primary and/or secondary users including various types of auctions [11]–[17] and pricing schemes [18]–[24]. As noted above, here we do not consider an explicit mechanism but instead focus on the problem of finding an efficient spectrum allocation. This can be viewed as a key part of a mechanism such as a VCG auction, in which the efficient allocation is determined from agents' submitted valuations.[4] The interference complementarities we study do not arise in most of the prior work because either the focus is on allocating spectrum at a single location, or it is assumed that no two interfering locations are allocated.

II. Basic Model for Spectrum Markets

Our basic model is for a market with a fixed set, C, of available *spectrum assets*, where each asset $j \in C$ represents the right to exclusively transmit with a fixed power mask over a given frequency band within a given geographic area.[5] We assume $|C|$ is large so that there are many assets to be allocated, and these assets are small enough relative to the given power mask that interference effects among them are significant.

Let A be the set of agents who wish to acquire the assets, and let $G = (C, E)$ be an *interference graph*, in which the set of directed edges, E, corresponds to pairs of interfering assets. Each agent here needs not be a single transmitter/receiver pair, but more generally can be a service provider that may seek to dynamically acquire spectrum at multiple locations to serve its customers. We assume that G is planar, as would be the case for interference due to spatial proximity. Let r_{ij} denote the revenue that agent i accrues when assigned asset j if there is no interference from any asset j' such that $(j, j') \in E$.[6] For example, r_{ij} could be proportional to the number of end users agent i serves in asset j. Furthermore, if agent i is assigned asset j and agent $q \neq i$ is assigned asset j' with $(j, j') \in E$, then agent i suffers an *interference cost* of $c^i_{jj'}$ and agent q suffers an interference cost of $c^q_{j'j}$ (assuming $(j', j) \in E$). Note that $c^i_{jj'}$ needs not be equal to $c^i_{j'j}$. We assume that $r_{ij} \geq \sum_{j':(j,j') \in E} c^i_{jj'}$ for all i and j, so that an agent never receives a negative utility (revenue minus costs) from an asset. If agent i acquires both assets j and j', she will not suffer this interference cost, due to the complementarity between assets. (Fig. 1 shows an example scenario for an agent $k \in A$ and three assets ($C = \{1, 2, 3\}$).) This complementarity could be due to reducing power in one asset, coordinating transmission schedules across assets or utilizing some type of cooperative transmission scheme. For now we do not focus on any particular underlying cause, but will return to this in Sect. IV.[7]

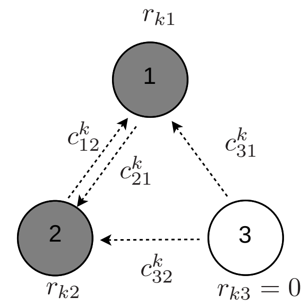

Fig. 1. Agent k's revenue and interference for three spectrum assets, $C = \{1, 2, 3\}$. Each shaded node denotes the asset for which agent k has positive revenue. The dashed arrows denote the interference cost agent k would incur without having the corresponding neighboring asset.

This model allows for multiple frequency bands at any location, where each band corresponds to a distinct asset. Assuming no interference across different bands, the interference graph consists of a separate component for each band.[8] We also assume that an agent's utility from acquiring multiple bands at a single location is simply the sum of the utility for each band. This is reasonable if an agent is serving users that are tied to a given band or has sufficiently many users to utilize all bands, but precludes cases where different bands are substitutes (e.g., where an agent desires one of two bands but not both). With these assumptions, the problem decomposes

[4] In a VCG auction, a version of the efficient allocation problem would also have to be solved for each agent to determine their payment.

[5] This definition is motivated in part by the discussion in [25].

[6] We assume that spectrum is scarce enough so that if agent i does not acquire it, then another agent will.

[7] Of course this linear model is a simplification. More elaborate models could be developed based on specific assumptions about how agents coordinate the use of neighboring assets. Even in such cases, agents could be restricted to report valuations in this linear form to simplify the market design.

[8] The model can be extended to allow interference across different bands modeling for example out-of-band interference due to different choices of receive filters. However, in this case, the resulting interference graph may not be planar and some of the following analysis would need to be modified.

into a separate problem for each band. Hence, we assume a single band in the following.

A. *Efficient Allocations*

A desirable goal for a market is to maximize efficiency (revenue minus cost). For our basic model, this is given by the following integer program:

$$\max \sum_{i\in A}\sum_{j\in C} r_{ij}x_{ij} - \sum_{i\in A}\sum_{(j,j')\in E} c^i_{jj'}(x_{ij}-x_{ij'})^+ \quad \text{(P1)}$$
$$\text{s.t.} \sum_{i\in A} x_{ij} \le 1,\ x_{ij}\in\{0,1\}, \forall j\in C, \forall i\in A, j\in C$$

where $x_{ij}=1$ if agent $i\in A$ is assigned asset $j\in C$ and zero otherwise.

Note that if there are no complementarities (i.e., $c^i_{jj'}=0$ for all $i\in A$ and $(j,j')\in E$), then (P1) is easy to solve; simply give each asset j to the agent with the largest value of r_{ij}. We next consider the complexity of (P1) when $c^i_{jj'}>0$.

B. *Computational Complexity*

By choosing large interference costs, one can ensure that when an agent is assigned an asset, no neighboring assets will be assigned to another agent. Using this idea, one can map the independent set problem into an instance of (P1), showing that it is NP hard. Moreover, as the next proposition states, the problem remains hard even for small interference costs.

Proposition 1: Problem (P1) is NP-hard even if the interference costs on each link is arbitrarily small (relative to the revenue).

The proof of this proposition is given in Appendix A.

The hardness in (P1) comes from the integer constraint. Indeed, relaxing this constraint to $0\le x_{ij}\le 1$ yields a linear program (LP) that is easily solvable. This LP will typically have fractional solutions, which cannot necessarily be interpreted as frequency or time sharing due to the need to coordinate such fractional assignments across assets (see [8]).

We next identify several scenarios in which (P1) can be efficiently solved.

1) Dominant Revenues: First we consider restrictions on the costs and revenues. We say that an agent i has a *dominant revenue* for an asset j if

$$r_{ij} \ge \sum_{j':(j,j')\in E} c^i_{jj'} + \sum_{j':(j,j')\in E} \max_{k\in A} c^k_{j'j}, \quad (1)$$

and i's net revenue (assuming interference from all neighbors, i.e., $r_{ij}-\sum_{j':(j,j')\in E} c^i_{jj'}$) is the largest among agents i' that also satisfy (1) with i' replacing i.

Proposition 2: Assume for each $j\in C$, there is at least one agent $i^*(j)$ with dominant revenue. Then under either of the following conditions, the optimal solution is to assign each asset j to $i^*(j)$: (i) for each $j\in C$ there is only one agent with positive revenue or (ii) each $i\in A$ has positive revenue for at most one asset.

The proof of this follows from showing that in each case, one can always improve the revenue by assigning an asset to an agent with a dominant revenue.

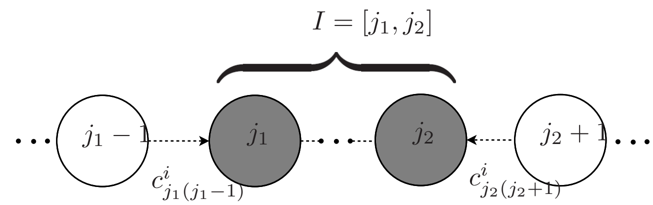

Fig. 2. An example of the line topology with interval $I=[j_1,j_2]$ for an agent i.

2) Line Model: In this scenario we restrict the topology of the interference graph to be a line with the assets numbered consecutively (see Fig. 2). We reformulate the optimization in terms of *intervals* of consecutive assets. Let $\mathcal{I}$ be the set of all intervals on the line, and $u_i(I)$ be the utility agent i receives from being allocated interval I and not any neighboring assets, i.e., $u_i(I)=\sum_{j\in I} r_{ij} - c^i_{j_1(j_1-1)} - c^i_{j_2(j_2+1)}$, where $I=[j_1,j_2]$. Problem (P1) can then be reformulated as follows, where x_{iI} indicates if interval I is assigned to agent i,

$$\max \sum_{i\in A}\sum_{I\in\mathcal{I}} u_i(I)x_{iI} \quad (\text{P}_l)$$
$$\text{s.t.} \sum_{i\in A}\sum_{I\ni j} x_{iI}\le 1 \text{ and } x_{iI}\in\{0,1\}\ \forall i\in A, I\in\mathcal{I}.$$

The next lemma shows that Problem (P$_l$) can be efficiently solved by linear programming.

Lemma 1: If the integer constraint in Problem (P$_l$) is relaxed to $0\le x_{iI}\le 1$, for all $i\in A, I\in\mathcal{I}$, then the resulting feasible set is a polyhedron with integral extreme points.

This lemma holds simply because the relaxed constraint matrix in (P$_l$) has consecutive 1's in each column and thus is *totally unimodular* [31].

3) Ring Model: The analysis of a line model can be generalized to a ring. In this case the corresponding reformulation is not totally unimodular, but is "nearly" so. This can be exploited to efficiently find a solution. The details are omitted due to space considerations.

The line and the ring cases suggest that one way to manage the market complexity would be to have lines or rings of assets separated by spatial guard zones, each such line or ring could be operated as a separate market in which prices are announced for intervals of assets as suggested by (P$_l$).

C. *Approximation Algorithms*

We next consider approximation algorithms for a general instance of Problem (P1). These are based on reformulating the problem by replacing the $(x_{ij}-x_{ij'})^+$ terms in the objective with $x_{ij}(1-x_{ij'})$ and introducing the new variables $z^i_{jj'} := x_{ij}x_{ij'}$ yielding

$$\max \sum_{i\in A}\sum_{j\in C}\tilde{r}_{ij}x_{ij} + \sum_{i\in A}\sum_{(j,j')\in\tilde{E}}\tilde{c}^i_{jj'}z^i_{jj'} \quad \text{(P2)}$$
$$\text{s.t.} \sum_{i\in A} x_{ij}\le 1,\ \ x_{ij}\in\{0,1\}, \forall i\in A, j\in C,$$
$$z^i_{jj'}\le x_{ij},\ z^i_{jj'}\le x_{ij'},\ \forall i\in A, (j,j')\in\tilde{E}$$

where $\tilde{r}_{ij}=(r_{ij}-\sum_{j':(j,j')\in E} c^i_{jj'})$, $\tilde{c}^i_{jj'}=c^i_{jj'}+c^i_{j'j}$ and $\tilde{E}$ is the set of *undirected* edges for the interference graph G

formed by replacing all directed edges between a pair of nodes by a single undirected edge. Here $\tilde{r}_{ij}$ is the minimum revenue agent i can gain from asset j (assuming interference from all neighbors), and $\tilde{c}^i_{jj'}$ is the *extra revenue* gained if i receives j and j' (or edge (j,j')). Let Z_{opt} denote the optimal value of (P2) or equivalently (P1).

1) Max-$\tilde{r}_{ij}$ approximation: First we consider a simple scheme: allocate each asset to the agent with the largest value of $\tilde{r}_{ij}$. The performance of this scheme is bounded as follows:

Proposition 3: Let $\gamma > 0$ be a constant so that $\sum_{j':(j,j')\in E} c^i_{jj'} \leq \gamma \tilde{r}_{ij}, \forall i \in A, j \in C$. Then the Max-$\tilde{r}_{ij}$ scheme gives a $(1+\gamma)$-approximation to (P2).

Proof: Let $\hat{Z}$ be the total utility achieved by this algorithm and let $i^*(j)$ be the agent assigned asset j in this solution. Note that $\hat{Z} \geq Z|_{\tilde{c}=0} = \sum_{j\in C} \tilde{r}_{i^*(j),j}$, where $Z|_{\tilde{c}=0}$ is the solution to a modified version of (P2) in which each of the $\tilde{c}^i_{jj'}$ terms is set to zero. Similarly, let $Z|_{\tilde{r}=0}$ be the solution to (P2) in which all of the $\tilde{r}_{ij}$ terms are set to zero, and let $\hat{i}(j)$ be the agent assigned to asset j in this solution. Note that $Z|_{\tilde{c}=0} + Z|_{\tilde{r}=0} \geq Z_{opt}$. By the definition of γ,

$$Z|_{\tilde{r}=0} \leq \sum_{j\in C} \sum_{j':(j,j')\in E} c^{\hat{i}(j)}_{jj'} \leq \sum_{j\in C} \gamma \tilde{r}_{\hat{i}(j),j} \leq \gamma Z|_{\tilde{c}=0}.$$

Combining we have $(1+\gamma)\hat{Z} \geq Z_{opt}$. ■

2) Max-r_{ij}: A related approximation is to assign each asset to the agent with the largest value of r_{ij}. By a similar proof this scheme has the following approximation bound:

Proposition 4: Let $\gamma' > 0$ be a constant so that $\sum_{j':(j,j')\in E} c^i_{jj'} \leq \gamma' r_{ij}$, for all $i \in A, j \in C$. Then allocating each asset to the agent with the largest r_{ij} gives a $1/(1+\gamma')$-approximation.

3) Edge coloring approximation: Consider a proper edge coloring of $\tilde{G} = (C, \tilde{E})$, which divides $\tilde{E}$ into q disjoint sets $E_1, ..., E_q$, one for each color. The objective function in (P2) can then be written as

$$Z = \sum_{j\in C} \sum_{i\in A} \tilde{r}_{ij} x_{ij} + \sum_{k=1}^{q} \sum_{jj'\in E_k} \sum_{i\in A} \tilde{c}^i_{jj'} z^i_{jj'}. \qquad (2)$$

Consider solving the following $q+1$ modified problems: for each $k = 1, \ldots, q$, one problem is given by replacing the objective in (2) with $\sum_{(j,j')\in E_k} \sum_{i\in A} \tilde{c}^i_{jj'} z^i_{jj'}$, and the final problem is given by replacing the objective in (2) with $\sum_{j\in C} \sum_{i\in A} \tilde{r}_{ij} x_{ij}$. Let Z_k denote the optimal value of each of these problems. Note that Z_{q+1} is equivalent to $Z|_{\tilde{c}=0}$ as defined for the previous approximation and so can be easily solved. Furthermore, the first q problems can also be solved by a greedy assignment of edges, since no two adjacent edges appear in their objectives. We then use the allocation for the problem with the largest value Z_k as our approximation.

Proposition 5: If a proper edge coloring of G can be found using q colors, then the preceding procedure gives a $(1+q)$-approximation.

Proof: Clearly, the allocation achieving $\max_k Z_k$ is also a feasible solution to (P2) and from (2) we have $Z_{opt} \leq \sum_{k=1}^{q+1} Z_k$. Hence, $\max_k Z_k \geq \frac{Z_{opt}}{q+1}$. ■

This approximation factor is minimized by setting q equal to the chromatic index χ of G. For a general graph determining χ is NP-complete, but it can be approximated to within 1 by the maximum degree D plus one. An edge coloring using $D+1$ colors can be easily found, resulting in a $(2+D)$-approximation.[9]

4) GRA-approximation: Let $Z|_{\tilde{c}=0}$ and $Z|_{\tilde{r}=0}$ be defined as in the Max-$\tilde{r}_{ij}$ approximation. Since $Z|_{\tilde{c}=0} + Z|_{\tilde{r}=0} \geq Z_{opt}$, it follows that either $Z|_{\tilde{c}=0} \geq 1/2 Z_{opt}$ or $Z|_{\tilde{r}=0} \geq 1/2 Z_{opt}$. As we have noted previously, computing $Z_{\tilde{c}=0}$ is easy. However, exactly computing $Z|_{\tilde{r}=0}$ is difficult in general. Indeed, by a similar argument as in the proof of Proposition 1, it can be shown that this is NP-hard. Instead we consider approximating this by adapting the *Geometric Rounding Algorithm* (GRA) in [26]. This involves solving the natural LP relaxation to (P2) and then applying a randomized dependent rounding scheme to get an integer solution. The specific scheme in [26] is shown to give a constant factor approximation to the *Winner Determination Problem* (WDP) in a combinatorial auction with single-minded bidders. The WDP is to efficiently allocate a collection of distinct goods to a set of bidders, where each bidder only desires a specific subset of the goods. The approximation factor for the GRA scheme in [26] is equal to the maximum cardinality of the subset desired by any agent.

Finding $Z|_{\tilde{r}=0}$ can be viewed as a generalization of the WDP problem in which the goods are assets to be allocated to each agent. Each agent i will only value pairs of goods (j,j') for which $c^i_{jj'} > 0$. However, in our case, agents are not single minded and may value multiple pairs, with an additive valuation across pairs. It can be seen that the results in [26] still apply with such a generalization, i.e., applying the GRA algorithm approximates $Z|_{\tilde{r}=0}$ with an approximation factor equal to the maximum cardinality of a subset desired by an agent (2 in our case). Using this we have the following bound.

Proposition 6: Taking the minimum of $Z|_{\tilde{c}=0}$ and the GRA approximation to $Z|_{\tilde{r}=0}$ is a 4-approximation.

We briefly comment on the implications of these approximations in terms of market mechanisms. The Max-$\tilde{r}_{ij}$ and the Max-r_{ij} only base their allocations on one value per agent for each asset and so would suggest a mechanism with lower overhead, and moreover, given these values, the allocation for each asset can be done separately. The challenge here is how to incentivize the agents to correctly report these values. For example, it is well known that using such an approximation in a truthful mechanism, like the VCG auction, can result in a mechanism that is no longer truthful. The edge coloring and GRA approximation require agents to report their full valuations for all bundles of assets and again ensuring that this is done truthfully is an open question.

D. Numerical Example

We present a numerical example to illustrate the performance of the preceding approximations for a square lattice with $|C| = 9$ assets and $|A| = 6$ agents. An agent's revenue for an asset is proportional to the number of end users within the asset, which are distributed according to a spatial Poisson

[9]Moreover, for certain graphs of interest such as regular lattices, χ is equal to the degree and a χ-edge coloring can be easily found giving a $(\chi+1)$-approximation.

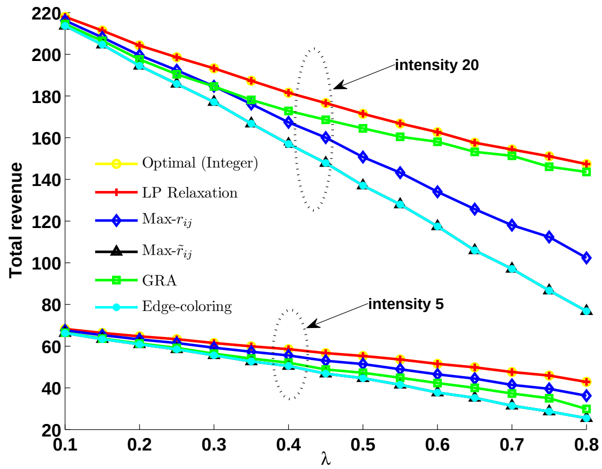

Fig. 3. Total revenue achieved by various approximation algorithms for a 3×3 lattice with six agents, and Poisson intensities of 5 and 20 users/asset.

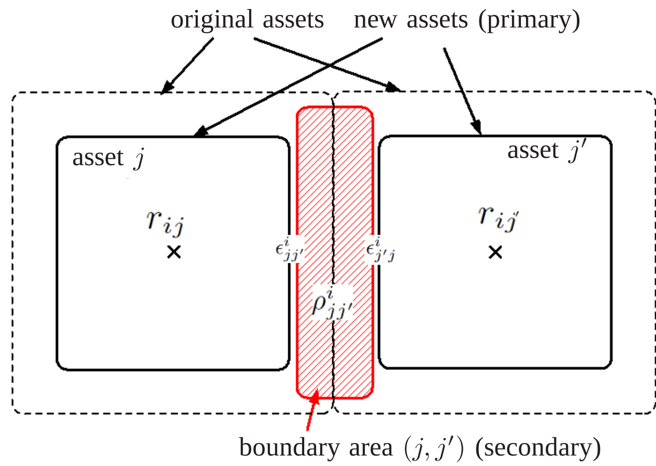

Fig. 4. Two adjacent assets showing the asset boundary area (shaded). The dashed lines represent the original asset boundaries and the solid lines represent new smaller asset boundaries.

process with intensity μ. The area of each asset is normalized to one, and the locations of end-user groups are independent. The interference cost is due to the inability to serve end users close to the asset boundary, modeled as the outer λ fraction of the asset's area. The interference cost is then proportional to the number of end users located within the corresponding boundary area.

Fig. 3 shows total revenue versus the interference area λ for the different approximations with intensities of $\mu = 5$ and $\mu = 20$ users per asset for all agents. These are compared with the optimal total revenue obtained from the integer solution to (P1). The revenue of a natural linear relaxation to (P1), which is an upper bound for the optimal revenue, is also shown in Fig. 3. Each point is an average over 500 realizations of the locations of the end users (and thus the revenues and costs).

The approximation algorithms achieve close to the maximum revenue for small λ. The gap widens as both λ and the spatial intensity μ increase, with the GRA-approximiation performing best for large λ and μ. However, for smaller μ, the Max-r_{ij} approximation performs best. This algorithm assigns assets based on their total value assuming no interference, while the other algorithms make assignments based on either the total revenue from the assets' boundary areas or interference-free areas. For $\lambda > 0.5$, the probability that an agent has more revenue in the boundary areas than in the interference-free area is increasing with μ. Hence for large μ an algorithm that focuses on the boundary areas such as the GRA-approximation performs better, while for smaller μ, the Max-r_{ij}, which accounts for the entire asset performs better.

III. Markets with Secondary Asset-Edge Users

In Section II, we have shown that a generic definition for spectrum assets with complementarities leads to an efficient allocation problem that is difficult to solve. In this section, we give an alternative model for spectrum assets that uses *secondary* agents in an attempt to improve on this complexity.

Interference primarily affects the users near asset boundaries, while users near the interior of an asset may receive little interference. Hence, we consider treating the asset boundary areas as separate assets from the interiors, which are restricted to be used by secondary agents that provide local service within those regions with lower Quality of Service (QoS), provided that they do not interferer with primary users. This is motivated by the well known model for secondary usage such as that proposed for the TV white-spaces [27]; however, here the secondary sharing is restricted to boundary regions which are allocated via a spectrum market. We show that allowing secondary boundary assets does not fundamentally reduce the complexity of the resulting efficient allocation problem provided that agents can operate in both a primary and secondary role.

This model is illustrated in Fig. 4, which shows two adjacent assets[10]. In our original model, these assets might be represented by adjacent spatial regions (dashed lines in Fig. 4). Now, we reduce the size of these regions, to create primary assets (solid lines in Fig. 4), that do not create significant interference with each other.[11] The area along the original asset boundaries (shaded in Fig. 4) becomes available for secondary use. Both the boundary and primary assets are allocated to any agent via a spectrum market.

As before, the set of primary assets, C, are related via an un-directed graph $G = (C, \tilde{E})$, where $\tilde{E}$ now represents assets that share a boundary area, i.e., each boundary area is indexed by an edge in $\tilde{E}$. A boundary area $(j, j') \in \tilde{E}$ only experiences interference from the neighboring assets j and j'.

Agent i receives revenue r_{ij} when assigned asset j regardless of whether or not she is assigned the neighboring boundary areas. Let $\rho^i_{jj'}$ denote the revenue agent i receives from boundary area (j, j') in isolation. If the agent owns asset j and the neighboring boundary area (j, j'), then the agent receives an additional (complementary) revenue of $\epsilon^i_{jj'}$. This is again due to the possibility of mitigating interference by coordinating transmissions across the asset and boundary area.

[10]The example in Fig. 4 uses square shape for assets for illustrative purpose only. Our model in this section applies for general interference graphs.

[11]Of course, the needed reduction will depend on the power masks used within each asset.

The efficient allocation is again given by an integer program with the objective:

$$\max \sum_{i\in A}\sum_{j\in C} r_{ij}x_{ij} + \sum_{i\in A}\sum_{(j,j')\in\tilde{E}} \rho^i_{jj'}y^i_{jj'} + \sum_{i\in A}\sum_{(j,j')\in\tilde{E}} (\epsilon^i_{jj'}z^i_{jj'} + \epsilon^i_{j'j}z^i_{j'j}). \qquad \text{(P3)}$$

This is optimized over the binary variables $\{x_{ij}, y^i_{jj'}, z^i_{jj'}\}$, where $x_{ij} = 1$ if agent i is assigned asset j, and is zero otherwise, $y^i_{jj'} = 1$ if the boundary area between assets j and j' is assigned to agent i , and $z^i_{jj'} = 1$ if agent i is assigned both asset j and the boundary (j, j'). Note that $z^i_{jj'}$ and $z^i_{j'j}$ refer to different assets. These are subject to analogous constraints as in (P2).

Next we show that (P3) is equivalent to a special case of (P2). Given an instance of (P3) with a graph G, construct a new graph G' which has a additional node for each node in G plus a node for each boundary area (edge) in G[12]; G' will have an edge between each node and the corresponding boundary area. By appropriately defining the costs for G', this equivalency follows. The resulting graph G' will have a special structure not present in a general graph G. Unfortunately, this structure does not make the problem more tractable. Indeed, using similar arguments as in Section II-B it can be shown that (P3) is still NP-hard. It also follows that the approximation algorithms in Sect. II-C apply to (P3). We can use the structure in (P3) to give an alternative approximation result for the Max-$\tilde{r}_{ij}$ algorithm. Specifically, if there is a constant γ' such that $\sum_{j':(j,j')\in E}\epsilon^i_{jj'} < \gamma'\left(r_{ij} + \frac{1}{2}\sum_{j':(j,j')\in E}\rho^i_{jj'}\right)$ then assigning each asset or boundary area to the agent with the largest value of r_{ij} or $\rho^i_{jj'}$ gives a $(1+\gamma')$-approximation. The constant γ' would likely be smaller than the corresponding γ for (P2).

In the above discussion, any agent can acquire the boundary areas. However, if assets can only be acquired by primary agents while boundary areas can only be acquired by a different group of secondary agents, then there will be no complementarities, and (P3) can be solved easily. Likewise, in such a setting designing a market mechanism is simple. Each assets and boundary region can essentially be sold via a separate mechanism, such as a posted price or a second-price auction.

The structure in (P3) can also be used to give a condition under which the problem can be solved by a simple greedy procedure. The precise statement follows.

Proposition 7: If for any boundary area $(j, j') \in \tilde{E}$, there exists some agent i such that $\rho^i_{jj'} \geq \rho^{i'}_{jj'} + \epsilon^{i'}_{jj'} + \epsilon^{i'}_{j'j}$ for all $i' \neq i$, then (P3) can be solved by a two-stage greedy algorithm that first assigns each boundary area to the agent with the largest value of $\rho^i_{jj'}$ and then assigns each asset to the agent with the largest revenue given the boundary assignment.

Proof: Consider using a two-stage dynamic programing to solve (P3): in the second stage, given an assignment of assets, optimally assign the boundary areas, and in the first stage assign the assets to maximize the current revenue plus the future revenue of the boundary assignment. Under the given condition, stage two always assigns each boundary area to the agent with the largest value of $\rho^i_{jj'}$, giving the stated algorithm. ■

[12]Namely, G' includes every node of G, but also introduces a new node for each boundary area.

IV. Markets with Flexible Asset Boundaries

So far we have assumed that the interference costs are given constants, which could model a variety of scenarios. We now consider a specific model for mitigating interference: adjusting the "radius" over which an agent can transmit in a given asset. For example, if agents serve users in each asset via downlink transmissions from a single access point, this can be accomplished by adjusting the access point's transmission power.[13] In other cases, such as uplink transmissions, the model can be viewed as simply determining the radius within which users may transmit (determined for example via GPS). We study market mechanisms that assign both assets and the radii, and show that this additional flexibility can reduce the complexity of the allocation problem.

For simplicity, we assume that the underlying (undirectional) interference graph is a *square* lattice; however, all of our problem formulations can be extended to other regular lattices (such as a hexagonal lattice) and our main results regarding complexity remain the same.

A. Model with Sectorization

We first consider a model in which each square asset is partitioned into four 90-degree sectors as shown in Fig. 5. For example, an access point in the center of each asset could use directional antennas to independently adjust the radius of each sector. The length of an asset is L, which is also the distance between the centers of neighboring assets. We assume that each sector experiences interference from only the closest sector in the neighboring asset. Each edge $(j, j') \in \tilde{E}$ corresponds to a pair of interfering sectors; we abuse notation and denote the corresponding sector in asset j (or j') by jj' (or $j'j$). Let $R^i_{jj'} \in [0, L/2]$ be the *radius* of agent i in sector jj', which is the minimal distance from the asset center to its boundary over which agent i can serve customers without interference. Let w_{ij} be the revenue per unit area of agent i in asset j, e.g., the density of agent i's customers in the asset. Agent i's revenue from sector jj' is then $w_{ij}R^{i2}_{jj'}$ in the absence of interference.

Interference costs are modeled by using an *interference boundary*, which extends beyond a sector's given radius by Δ units.[14] Specifically, an interference cost is incurred in both sectors jj' (assigned to i) and jj' (assigned to k) when $R^i_{jj'} + R^k_{j'j} \geq L - \Delta$. Interference from asset j in sector $j'j$ can be ignored beyond distance $R^i_{jj'} + \Delta$ from the center of asset j. Let $z^{ik}_{jj'} = \max\{R^i_{jj'} + R^k_{j'j} - (L - \Delta), 0\}$ denote the

[13]Of course in practice, exactly controlling the "radius" of transmission is not possible due to effects such as fading. This quantity is better viewed in an average sense over the relevant time-scale at which allocations are performed.

[14]This is similar to the interference footprint in the standard protocol model from [28]. Of course, this is an over-simplification; but it provides a first order model of how one can adapt the interference externality.

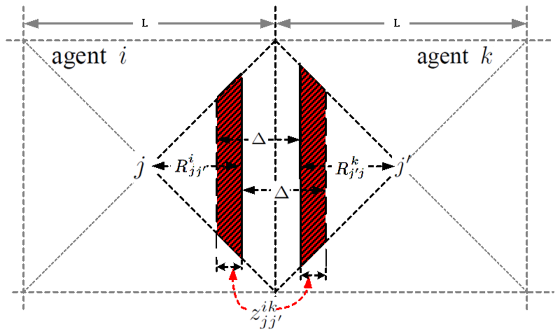

Fig. 5. Illustration of the radii model for two adjacent square assets each with four sectors.

amount of overlap of interference boundaries in sectors jj' and $j'j$. Agents receive no revenue for any area within this overlap (the shaded area in Fig. 5). Hence, the revenue of agent i in sector jj' is $w_{ij}(R^i_{jj'} - z^{ik}_{jj'})^2$. No additional interference management is assumed between sectors assigned to the same agent, i.e., $z^{ii}_{jj'}$ is not necessarily zero. Interference among the sectors assigned to the same agent is managed by the market optimizing the radii.

The efficient allocation is given by the following *mixed-integer quadratic program* (MIQP) :

$$\max_{\mathbf{x},\mathbf{R},\mathbf{z}} \sum_{i\in A}\sum_{j\in C}\sum_{j':(j,j')\in E} w_{ij}(R^i_{jj'} - \sum_{k\in A} z^{ik}_{jj'})^2 \quad \text{(P4)}$$

$$\text{s.t. } x_{ij}(\frac{L}{2} - \Delta) \le R^i_{jj'} \le x_{ij}\frac{L}{2}, \forall i \in A, (j,j') \in E$$

$$0 \le R^i_{jj'} + R^k_{j'j} - z^{ik}_{jj'} \le L - \Delta, \quad \forall i,k \in A, (j,j') \in \tilde{E}$$

$$\sum_i x_{ij} \le 1, \; z^{ik}_{jj'} \ge 0, \; x_{ij} \in \{0,1\} \quad \forall i,k \in A, \; j \in C, \; (j,j') \in \tilde{E}.$$

This is clearly a simplified model that we have chosen to highlight the potential advantages of having a market determine both asset assignments and radii. We briefly comment on a few of these simplifications. The assumption that assets are located on a regular lattice is one simplification; this could be relaxed for example by introducing different distances $L_{jj'}$ for different pairs of assets. Another simplification is the model for interference costs; one could use a more sophisticated physical layer model to capture these effects and these costs could vary among providers who use different technologies and/or have different QoS requirements. Finally, the revenue model does not account for capacity constraints, which could preclude a provider from serving all users within an asset.

Problem (P4) can be solved via a two-step procedure: (i) determine an assignment of assets to agents and (ii) determine the radii of each sector for the assigned assets. The following lemma shows that the first step can be solved independently of the second.

Lemma 2: In an optimal solution to (P4), $x^*_{ij} = 1$ if and only if $i = \arg\max_i w_{ij}$ for each $j \in C$.

Proof: Let $\{\tilde{x}_{ij}, \tilde{R}^i{}_{jj'}, \tilde{z}^{ik}_{jj'}\}$ be an optimal solution to (P4) and suppose that the lemma is not true for some asset j. Let $\tilde{i}$ be the user currently assigned asset j and let $i^* = \arg\max_i w_{ij}$. Re-assigning asset j to agent i^* with the same radii for each sector and the same choices of $z^{ik}_{jj'}$ and keeping all other variables unchanged must still be a feasible solution with the same area served in each asset. Moreover, the revenue from asset j will increase and so the original solution cannot be optimal. ∎

Given an optimal asset assignment, the optimal radii are as in Lemma 2, we next consider optimizing the asset radii. This is given by the following quadratic program (QP):

$$\max_{\mathbf{R},\mathbf{z}} \sum_{(j,j')\in\tilde{E}} w_j(R_{jj'} - z_{jj'})^2 + w_{j'}(R_{j'j} - z_{jj'})^2 \quad \text{(P5)}$$

$$\text{s.t. } 0 \le R_{jj'} + R_{j'j} - z_{jj'} \le L - \Delta, \forall (j,j') \in \tilde{E}$$

$$\frac{L}{2} - \Delta \le R_{jj'} \le \frac{L}{2}, \forall (j,j') \in E$$

$$z_{jj'} \ge 0, \; \forall (j,j') \in \tilde{E}$$

where we have dropped the agent indices, since the agent assigned to each asset is given. The objective of this QP is convex and so it cannot be solved directly by using first order conditions. However, its extreme points have the following useful property:

Lemma 3: An optimal solution to (P5) must satisfy $R^*_j \in \{L/2 - \Delta, (L - \Delta)/2, L/2\}$.

The proof is similar to the proof of Proposition 2.1 in [29] and so we omit it here.

Since each sector jj' only interferes with the neighboring sector $j'j$, (P5) can be separated into a collection of subproblems, one for each $(j,j') \in E$. The subproblem for $(j,j') \in E$ only involves the variables $R_{jj'}$, $R_{j'j}$ and $z_{jj'}$, which from Lemma 3 can take on only a finite number of values each. Hence, we can solve (P5) and thus (P4) in polynomial-time.

For the case of hexagonal assets with 120-degree sectorization, the efficient allocation problem is again a quadratic problem that is very similar to (P4). Thus, Lemmas 2 and 3 can be extended directly. The only slight difference is that the corresponding (P5) is separated into a collection of subproblems, one for each group of three interfering sectors. Therefore, the problem can still be solved in polynomial-time.

The key difference between (P4) and (P1) is due to letting the market assign the radii. Indeed, if the radii are not determined by the market, (P4) is equivalent to a special case of (P1), and is still NP-hard. Specifically, suppose that an agent always uses the maximum radius $L/2$ if it is assigned sector jj' but not sector $j'j$, so that $z^{ik}_{jj'} = \Delta$. An agent assigned both sectors jj' and $j'j$ can optimize both radii as in (P5). This maps to (P1) by letting r_{ij} be the revenue obtained from asset j using the optimal radii $\{R^{i*}_{jj'}\}_{j':(j,j')\in E}$ given by the solution to (P5) and letting $c^i_{jj'} = w_{ij}(R^{i*}_{jj'})^2 - w_{ij}(L/2-\Delta)^2$.

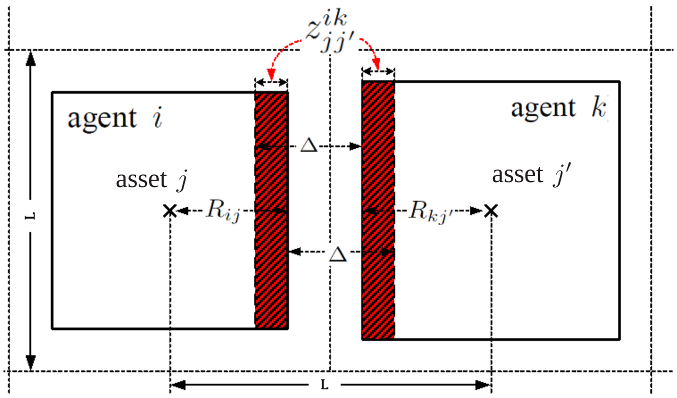

Fig. 6. Illustration of the omnidirectional radii model for a square lattice.

B. Omnidirectional Model

Next we consider a variation without sectorization, so that assets have the same radii in each direction. For example, this models a system in which agents transmit from the center of an asset using an omnidirectional antenna. With sectorization, the optimization of asset radii decomposes into a separate problem for each pair of interfering sectors. In an omnidirectional model, the optimization of asset radii becomes coupled across multiple assets. Nevertheless, we will show that a linearized version of this problem can still be efficiently solved.

We consider the same model as in the previous section except a single radius R_{ij} is used for asset j by agent i. Hence, the assets are squares as shown in Fig. 6. Again, the assumption of square assets is only for the ease of presentation. The formulation and results can be extended to hexagonal lattices in a straightforward way. Agent i's revenue when assigned asset j with radius R_{ij} is then $4w_{ij}R_{ij}^2$ minus the interference costs from any overlap with the interference footprint of neighboring assets (shaded area in Fig. 6). The revenue agent i receives from asset j is then given by

$$w_{ij}\left(2R_{ij} - \sum_{k\in A}(z^{ik}_{jj^n} + z^{ik}_{jj^s})\right) \times \left(2R_{ij} - \sum_{k\in A}(z^{ik}_{jj^w} + z^{ik}_{jj^e})\right) \tag{3}$$

where $z^{ik}_{jj'}$ again denotes the amount of overlap of a neighboring asset's interference area and j^n, j^s, j^w, and j^e denote the assets to the north, south, east and west of j (with respect to an arbitrary choice of north). The efficient allocation is then given by the following MIQP:

$$\begin{aligned}\max_{\mathbf{x},\mathbf{R},\mathbf{z}} \quad & \sum_{i\in A}\sum_{j\in C} w_{ij}\left(2R_{ij} - \sum_{k\in A}(z^{ik}_{jj^n} + z^{ik}_{jj^s})\right) \\ & \times \left(2R_{ij} - \sum_{k\in A}(z^{ik}_{jj^w} + z^{ik}_{jj^e})\right) \qquad \text{(P6)} \\ \text{s.t.} \quad & 0 \le R_{ij} + R_{kj'} - z^{ik}_{jj'} \le L - \Delta, \\ & \qquad \forall i,k\in A, (j,j')\in\tilde{E} \\ & x_{ij}(\frac{L}{2} - \Delta) \le R_{ij} \le x_{ij}\frac{L}{2}, \forall i\in A,\, j\in C \\ & \sum_i x_{ij} \le 1,\, z^{ik}_{jj'} \ge 0,\, x_{ij}\in\{0,1\}, \\ & \qquad \forall\, i,k\in A,\, j\in C,\, (j,j')\in\tilde{E}.\end{aligned}$$

Lemmas 2 and 3 can be generalized to this problem. However, given an assignment of assets, the resulting QP for determining the radii is now coupled across the assets and the objective is neither concave or convex, making this difficult to solve for a large number of assets. However, after making the assignment of assets, the number of remaining variables is much smaller; there will be no more than $3|C|$ variables while before making an assignment there are on the order of $2|A|^2|C| + |A||C|$ variables; hence, for a moderate number of assets, it is feasible to use a commercial solver to determine the optimal radii.[15] Alternatively, we next consider a linearized version of this problem which yields a more tractable solution.

Note that since $R_{ij} \le L/2$, no interference costs will be incurred if an agent uses a radius $L/2 - \Delta$. Thus, the revenue that an agent gains from an asset can be represented as the sum of the revenue from a square with radius $L/2 - \Delta$ and the remaining area, which may incur an interference cost. Specifically, by replacing R_{ij} in (3) with $(\frac{L}{2} - \Delta) + (R_{ij} - (\frac{L}{2} - \Delta))$ and simplifying the resulting expression, (3) can be rewritten as

$$\begin{aligned} & 8\left(\frac{L}{2} - \Delta\right)\left[w_{ij}R_{ij} - \sum_{j':(j,j')\in E}\frac{1}{4}w_{ij}\sum_{k\in A} z^{ik}_{jj'}\right] \\ & - 4w_{ij}\left(\frac{L}{2} - \Delta\right)^2 + O(\Delta^2) \end{aligned} \tag{4}$$

where we have used that $R_{ij} \le L/2$ and $z^{ik}_{jj'} \le \Delta$ to get the $O(\Delta^2)$ bound.

Dropping the $O(\Delta^2)$ terms, the remaining terms are linear in the optimization variables. Furthermore, we can drop the constant term $4w_{ij}(\frac{L}{2} - \Delta)^2$, and the (non-negative) scaling term of $8\left(\frac{L}{2} - \Delta\right)$, giving the following new optimization problem:

$$\begin{aligned}\max_{\mathbf{x},\mathbf{R},\mathbf{z}} & \sum_{i\in A}\sum_{j\in C}\left(w_{ij}R_{ij} - \sum_{j':(j,j')\in E}\sum_{k\in A}\alpha w_{ij} z^{ik}_{jj'}\right) \qquad \text{(P7)} \\ \text{s.t.} & (\mathbf{x},\mathbf{R},\mathbf{z}) \in \mathcal{P}\end{aligned}$$

where $\mathcal{P}$ denotes the same constraint set as in (P6). In fact, this linear approximation can be applied to any regular lattice with $\alpha = \frac{1}{4}$ for a square lattice as derived and $\alpha = \frac{1}{6}$ for a hexagonal lattice.

Lemma 2 can be extended to (P7) and so this problem can again be solved by first allocating each asset to the agent with the largest w_{ij} and then optimizing the radii, which is now a linear program, and so (P7) can be efficiently solved. Note that the optimal asset assignment is the same in both (P6) and (P7). Of course, if the true valuation is given by the objective in (P6), then solving this linear version will lead to a loss in revenue, which is characterized by the following lemma. Let $f(\mathbf{R})$ be the objective value of (P6) for some particular radii vector $\mathbf{R}$ that satisfies $\mathcal{P}$.

[15] Note that by using Lemma 3 this can be formulated as an integer QP.

Lemma 4: Suppose $\mathbf{R}^*$ and $\tilde{\mathbf{R}}$ are the solution to (P6) and (P7), respectively. Then, $f(\mathbf{R}^*) - f(\tilde{\mathbf{R}}) \le \beta \Delta L \sum_{j \in C} \max_{i \in A} w_{ij}$, where $\beta = 4$ for a square lattice and $\beta = 2\sqrt{3}$ for a hexgonal lattice.

The lemma can be proved simply by obtaining an upper and lower bound on $f(R^*)$ and $f(\tilde{R})$, respectively, and thus omitted. Lemma 4 suggests that this linear approximation is reasonable for small Δ. Numerical results, in the next section, show that the loss may be small even for large values of Δ.

C. Numerical Results

We present some numerical comparisons of the revenue achieved by the original model and the radii models with and without sectorization. A square lattice with 4×4 assets and 6 agents is used. We set $L = 1$ and for the radii model the w_{ij}'s are randomly generated following a Poisson distribution with intensity $\mu = 50$. As in Sect. II-D, $1 - \lambda$ denotes the fraction of each asset which is always interference free. For the radii models, this is equivalent to choosing $\Delta = (1 - \sqrt{1-\lambda})/2$. The revenues and costs for the original model are assigned as in Sect. IV-A. For the original model, we exactly solve (P1). For the omnidirectional model, we solve (P7) to determine the asset assignment and radii used, but then plot the corresponding revenue using the objective of (P6). We also solve (P6) numerically. As a benchmark, we also show results for a model with spatial guard zones, i.e., agents are required to use a radius of $\frac{L}{2} - \Delta$ so that no two assets interfere. In this case, the efficient allocation is to assign each asset j to the agent with the largest value of w_{ij}.

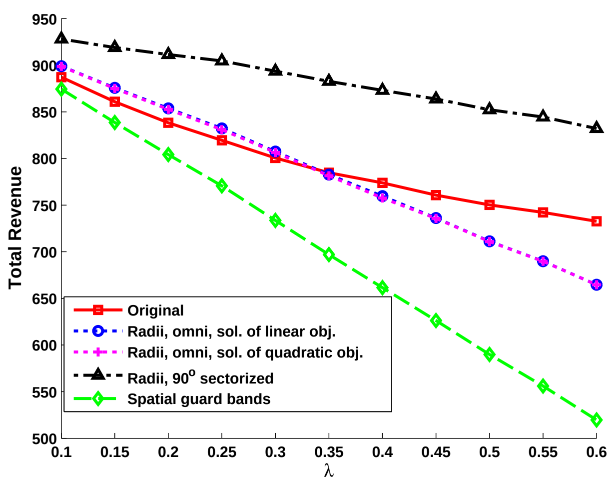

Fig. 7. The total revenue versus the amount of interference under different market models.

Fig. 7 shows the total revenue for each model versus λ, averaged over 200 realizations of the w_{ij}'s. The fixed guard zone model achieves the lowest revenue, which goes to zero as λ increases since no revenue is obtained in the guard zones. Even for moderate values of λ, the other approaches achieve significantly higher revenue, demonstrating the potential benefit of the spectrum market managing interference. The highest revenue is achieved by the radii model with sectorization which offers the most flexibility in assigning resources. Both curves for the omnidirectional model are indistinguishable for the entire range of λ, showing that at least for this scenario, the linear model is a good approximation. For small enough λ (equivalently small Δ), the omnidirectional model outperforms the original model. However, for large λ (large Δ), the omnidirectional model has lower revenue than the original model. While the omnidirectional model has the flexibility of optimizing asset radii across agents, the costs for the original model are based on allowing agents to adapt asset radii on a sector basis when they own neighboring assets. Hence, it is not clear that one of these schemes will always perform better than the other. For large λ, the revenue from "boundary regions" is greater and apparently the original model has better performance.

V. Conclusions

We have examined several simple models of spectrum markets with interference complementarities and shown how different market structures can impact both the computation of an efficient outcome and the resulting revenue. For a basic model in which the market specified only the assignment of assets to users, the resulting assignment problem was shown to be NP-hard, which suggests that "simple" mechanisms will not be able to obtain the efficient outcome in such markets. Several approximations were given, which had good performance in numerical examples. Next, we considered a market where guard zones between primary assets were allocated for secondary use. This did not improve the complexity of the general allocation problem, but provided structure that can be exploited in several cases. Finally, we considered models in which the market determined both the assignment of assets and their radii, which led to simpler allocation problems as well as higher total revenue. These examples illustrate just a few of the rich possibilities in defining spectrum assets and the non-trivial interactions between these definitions and the complexity and efficiency of the resulting market design.

Future research directions include developing more refined models for interference costs and studying their effect on the resulting markets, studying strategic behaviors of agents in such markets and developing mechanisms to implement the markets.

Appendix

A. Proof of Proposition 1

Given a graph $G = (V, E)$ and positive number $K \ge 3$, the *Graph Partitioning* problem is to find a partition of V in to disjoint sets V_1,...,V_m such that $|V_i| \le K$ for all $1 \le i \le m$ and such that if $E' \subset E$ is the set of edges that have two endpoints in two different sets, then $|E'|$ is minimized, where $|\cdot|$ is the cardinality of the set. (see [30] for a general version of this problem). This problem is NP-complete, even with the restriction that $K = 3$.

We give a transformation of the graph partitioning problem with $K = 3$ into the spectrum asset allocation problem. Let V be the set of spectrum assets and G the corresponding interference graph. For any $V_i \subset V$ such that $|V_i| \le 3$, introduce an agent with $r_{ij} = r_0$ only for $j \in V_i$, and zero otherwise; also, set $c^i_{jj'} = c_0$ for all edges $(j, j') \in E$ such that $j \in V_i$ or $j' \in V_i$ or both. The number of agents resulting from this $\frac{1}{6}n^3 + O(n)$. Thus, this transformation can be done in polynomial time. Furthermore, assume $r_0 > 0$ and that $c_0 > 0$ is small enough so that an agent's revenue is always greater than the costs she suffers, as well as the costs she imposes on the agents owning the neighboring assets. This can always be done and allows for c_0 to be arbitrarily small relative to r_0. As a result, each asset j will be allocated to some agent for which $r_{ij} = r_0$. Hence, the first term in the objective function in (P1) becomes a constant ($= |V|r_0$) and the optimal solution is that the one which minimizes the total interference costs among the assets. Since all interference costs are equal, the total interference cost of an assignment is the number of edges whose end nodes belong to different agents. By construction, the number of assets assigned to a given agent will be no greater than 3 and hence the solution to this also a solution to the graph partitioning problem.

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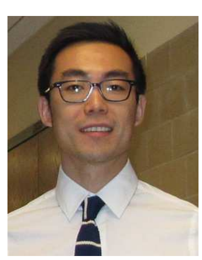

Hang Zhou received the B.S. degree in Electronic Engineering and Information Science from the University of Science and Technology of China (USTC) in 2005 and the M.S. and Ph.D. degrees in Electrical Engineering and Computer Science from Northwestern University in 2007 and 2012, respectively. In 2012 he joined the advanced modem team at Motorola Mobility as a senior engineer. His primary research interests include wireless communication, networking, optimization and network economics.

Randall A. Berry received the B.S. degree in Electrical Engineering from the University of Missouri-Rolla in 1993 and the M.S. and PhD degrees in Electrical Engineering and Computer Science from the Massachusetts Institute of Technology in 1996 and 2000, respectively. In 2000, he joined Northwestern University, where he is currently a Professor in the Department of Electrical Engineering and Computer Science. In 1998 he was on the technical staff at MIT Lincoln Laboratory in the Advanced Networks Group. His primary research interests include wireless communication, data networks, network economics and information theory.

Dr. Berry is the recipient of a 2003 NSF CAREER award. He has served as an Editor for the IEEE Transactions on Wireless Communications from 2006 to 2009, and an Associate Editor for the IEEE Transactions on Information Theory from 2009 to 2011, in the area of communication networks. He was a co-chair of the 2012 IEEE Communication Theory Workshop and a technical co-chair of 2010 IEEE ICC Wireless Networking Symposium. He has also been a guest editor for the IEEE Journal on Selected Topics in Signal Processing special issue on ”Dynamic Spectrum Access” and the IEEE Transactions on Information Theory special issue on ”Relaying and Cooperation.”

Michael L. Honig received the B.S. degree in electrical engineering from Stanford University in 1977, and the M.S. and Ph.D. degrees in electrical engineering from the University of California, Berkeley, in 1978 and 1981, respectively. He subsequently joined Bell Laboratories in Holmdel, NJ, where he worked on local area networks and voiceband data transmission. In 1983 he joined the Systems Principles Research Division at Bellcore, where he worked on Digital Subscriber Lines and wireless communications. Since the Fall of 1994, he has been with Northwestern University where he is a Professor in the Electrical and Computer Engineering Department. He has held visiting scholar positions at the Naval Research Laboratory (San Diego), the University of California, Berkeley, the University of Sydney, Princeton University, and the Technical University of Munich. He has also worked as a free-lance trombonist.

Dr. Honig has served as an editor for the IEEE Transactions on Information Theory (1998-2000) and the IEEE Transactions on Communications (1990-1995), and as guest editor for the Journal on Selected Topics in Signal Processing, European Transactions on Telecommunications, and Wireless Personal Communications. He has also served as a member of the Digital Signal Processing Technical Committee for the IEEE Signal Processing Society, and as a member of the Board of Governors for the Information Theory Society (1997-2002). He is a Fellow of IEEE, the recipient of a Humboldt Research Award for Senior U.S. Scientists, and the co-recipient of the 2002 IEEE Communications Society and Information Theory Society Joint Paper Award and the 2010 IEEE Marconi Prize Paper Award.

Rakesh Vohra was educated at University College London, the London School of Economics and the University of Maryland, where he received a Ph.D. in applied mathematics. He is presently the John L. and Helen Kellogg Professor of Managerial Economics and Decision Sciences at Northwesterns Kellogg School of Management. He also serves as the Director of the Center for Mathematical Studies in Economics and Management Science. His primary research interests are auction theory and game theory.