Spectrum Markets with Interference Complementarities

Hang Zhou*, Randall A. Berry*, Michael L. Honig * and Rakesh Vohra†
*EECS Department, †CMS-EMS, Northwestern University, Evanston, IL 60208
{hang.zhou, rberry, mh}@eecs.northwestern.edu, r-vohra@kellogg.northwestern.edu

***Abstract*—Extensive spectrum markets have the potential to enable more efficient use of this limited resource. Such markets must account for particular properties of the underlying wireless medium. In this paper we focus on one such aspect: the role of interference created among different agents who may purchase the right to use the same spectrum at nearby locations. Such interference can result in "complementarities" among the spectrum goods being traded, which complicates the design of an efficient market. We begin with a simple linear model for these complementarities that was shown to be computationally difficult in earlier work. We give several approximation algorithms for this model. We then consider several alternative models in which the spectrum goods are defined in different ways and explore the impact of these choices on the complexity of the resulting market.**

I. Introduction

It is widely recognized that current spectrum policy has resulted in inefficiencies and underutilization. One proposed solution to this is to establish markets to enable a more flexible allocation of spectrum [1], [2]. Provisions for limited forms of such spectrum markets have been adopted in the U.S. [3]. The design of spectrum markets must account for the fact that transmitting in the same spectrum at nearby locations creates interference, which differentiates spectrum from many other goods. Namely, an agent's value for spectrum at a particular location may, because of interference, depend on the use of the spectrum at nearby locations. Of course, one approach for dealing with such interference is to only allocate a given band of spectrum in two locations that are sufficiently far apart, in effect creating a "spatial guard zone" between these locations to avoid any interference. However, if spectrum is allocated on a small geographic scale, the overhead from such guard zones could become significant. Moreover, the acceptable interference at one location may vary greatly depending on the application and the underlying technology used.

Instead of avoiding interference between two spectrum assets, here we consider markets in which interference does occur but can be managed by agents purchasing neighboring assets to preclude other interfering agents from using them, effectively creating their own guard zones on demand. This results in complementarities in "bundles" of spectrum assets (i.e., the value of a bundle of assets may be greater than the sum of the values of the individual assets). In addition to creating guard zones, an agent could mitigate interference by coordinating transmissions across neighboring spectrum it owns (in frequency and/or space). Such approaches would again lead to complementarities.[1]

In [4] a simple linear model for allocating spectrum with interference complementarities was presented. Determining an efficient spectrum allocation for this model was shown to be NP-hard. Here, we first return to this model and give several approximation schemes. We then consider different models for allocating spectrum in such a market. In Sect. IV, we present a model that is based on using spatial guard zones, but allows secondary users to utilize these zones. In Sect. V, we present a model in which interference complementarities are related to a "radius" over which an agent may transmit. We show that allowing the market to specify this radius in addition to the assignment of spectrum simplifies the calculation of an efficient allocation.

In terms of related work, a number of papers have discussed mechanisms for allocating spectrum to primary and/or secondary users including various types of auctions [5]–[7] and pricing schemes [8]–[10]. Here, we do not consider an explicit mechanism but instead focus on the problem of finding an efficient spectrum allocation. This can be viewed as part of a mechanism such as a VCG auction, in which agents submit valuations for spectrum and the resulting problem is solved to determine the allocation.[2] We also note that the interference complementarities we study do not arise in most of the prior work because either the focus is on allocating spectrum at a single location, or it is assumed that no two interfering locations are allocated the same spectrum.

II. Basic Model for Spectrum Allocation

We start by reviewing the model from [4]. Let C denote the set of available *spectrum assets*, where each asset $j \in C$ represents the right to exclusively transmit with a fixed power mask over a given frequency band within a given geographic area.[3] We will also refer to assets as "cells" to emphasize that they correspond to specific geographic regions. We are

This research was supported in part by NSF under grants CNS-0519935 and CNS-0905407.

[1] We focus on complementarities due to interference but note that complementarities between adjacent spatial locations can exist for other reasons as well, e.g., supporting soft hand-offs in a CDMA network.

[2] In a VCG auction, a related problem would also have to be solved for each agent to determine their payment.

[3] This definition is motivated in part by the discussion in [11].

interested in the scenario where $|C|$ is large so that there are many such assets to be allocated, and these assets are small enough relative to the given power mask that interference effects among them are significant. This differs from most current spectrum auctions in which the assets allocated are large (e.g., on a national scale) so that interference between assets is generally negligible.

Let A be the set of agents who wish to acquire these assets and let $G = (C, E)$ be an *interference graph*, in which the set of directed edges E corresponds to pairs of interfering cells. We further assume that G is planar, as would be the case for interference due to spatial proximity. Let r_{ij} denote the revenue that agent i accrues when assigned asset j if there is no interference from any asset j' such that $(j, j') \in E$.[4] For example, if an agent is a service provider, then r_{ij} can be assumed to be proportional to the number of end users agent i serves in cell j. Furthermore, if agent i is assigned asset j and agent $q \neq i$ is assigned a neighboring asset j' with $(j, j') \in E$, then agent i suffers an *interference cost* of $c^i_{jj'}$ and agent q suffers an interference cost of $c^q_{j'j}$ (assuming $(j', j) \in E$). These costs may be directional so that $c^i_{jj'}$ need not be equal to $c^i_{j'j}$. We assume for each $i \in A$ and $j \in C$ that $r_{ij} \geq \sum_{j':(j,j')\in E} c^i_{jj'}$ so that an agent never receives a negative utility (revenue minus costs) from an asset. If agent i acquires both assets j and j', she will not suffer this interference cost. The reduction in interference costs models the complementarity between two neighboring cells as discussed in the introduction. This reduction could be due to a number of different causes, for example reducing power in one cell, coordinating transmission schedules across cells or utilizing some type of cooperative transmission scheme. For now we do not focus on any particular underlying cause, but will return to this in Sect. V.[5]

This model allows for multiple frequency bands at any given location, in which case each band would correspond to a distinct asset/cell. Assuming that there is no interference across different frequencies, the resulting interference graph consists of a separate component for each band.[6] Furthermore, we assume that the utility an agent receives from acquiring multiple bands at a single location is simply the sum of the utility for each band. This is is reasonable if an agent is serving users that are tied to a given band or has sufficiently many users to utilize all bands, but precludes cases where different bands are substitutes for each other (e.g. where an agent desires one of two bands but not both). With these assumptions it follows that a problem with multiple bands decomposes into a set of separate problems, one for each band. Hence, we assume a single frequency band in the following.

A. Efficient Allocations

Our interest is to find an efficient spectrum allocation, i.e., one that maximizes the total revenue minus costs summed over all agents. Let $x_{ij} = 1$ if agent $i \in A$ is assigned asset $j \in C$ and zero otherwise. The efficient allocation is then given by the following integer program:

$$\max \sum_{i\in A}\sum_{j\in C} r_{ij}x_{ij} - \sum_{i\in A}\sum_{(j,j')\in E} c^i_{jj'}(x_{ij} - x_{ij'})^+ \qquad \text{(P1)}$$
$$\text{s.t.} \sum_{i\in A} x_{ij} \leq 1,\ \forall j \in C \text{ and } x_{ij} \in \{0,1\},\ \forall i \in A, j \in C.$$

The objective function of this problem is concave but non-differentiable. Note that if there are no complementarities (i.e., $c^i_{jj'} = 0$ for all $i \in A$ and $(j, j') \in E$), then (P1) is easy to solve; one should simply give each asset j to the agent with the largest value of r_{ij}. However, with non-zero interference costs we have shown in [4] that this problem is NP hard. Moreover, the problem remains NP-hard even if the interference costs are arbitrarily small. Motivated by this we next consider approximation algorithms for this problem.[7]

III. Approximation Algorithms

Replacing the $(x_{ij} - x_{ij'})^+$ terms in the objective of (P1) with $x_{ij}(1 - x_{ij'})$ and introducing the new variables $z^i_{jj'} := x_{ij}x_{ij'}$ yields the following equivalent formulation:

$$\begin{aligned}
\max &\sum_{i\in A}\sum_{j\in C} \tilde{r}_{ij}x_{ij} + \sum_{i\in A}\sum_{(j,j')\in \tilde{E}} \tilde{c}^i_{jj'} z^i_{jj'} \qquad \text{(P2)}\\
\text{s.t.} &\sum_{i\in A} x_{ij} \leq 1 \quad \forall j \in C\\
& z^i_{jj'} \leq x_{ij},\ \forall i \in A, (j, j') \in \tilde{E}\\
& z^i_{jj'} \leq x_{ij'},\ \forall i \in A, (j, j') \in \tilde{E}\\
& x_{ij} \in \{0,1\} \quad \forall i \in A, j \in C,
\end{aligned}$$

where $\tilde{r}_{ij} = (r_{ij} - \sum_{j':(j,j')\in E} c^i_{jj'})$, $\tilde{c}^i_{jj'} = c^i_{jj'} + c^i_{j'j}$ and $\tilde{E}$ is the corresponding set of *undirected* edges for the interference graph G formed by replacing all directed edges between a pair of nodes by a single undirected edge. In this formulation $\tilde{r}_{ij}$ can be viewed as the minimum revenue agent i can gain from asset j (assuming she suffers interference from all neighboring assets), and $\tilde{c}^i_{jj'}$ can be viewed as the *extra revenue* gained if agent i receives two complementary assets j and j' (or edge (j, j')). Let Z_{opt} denote the optimal value of this problem (or equivalently (P1)). Next we use this reformulation to develop several approximation schemes.

[4] We assume that spectrum is scarce enough so that if agent i does not acquire it, then another agent will.

[5] Of course this linear model is a simplification. More elaborate models could be developed based on specific assumptions about how agents coordinate the use of neighboring assets. Even in such cases, agents could be restricted to report valuations in this linear form to simplify the market design.

[6] The model can be extended to allow interference across different bands modeling for example out-of-band interference due to different choices of receive filters. However, in this case, the resulting interference graph may not be planar and some of the following analysis would need to be modified.

[7] In the case of a VCG mechanism, if the market uses such an approximation the mechanism is no longer guaranteed to be incentive compatible. We do not address such incentive issues here.

1) Max-$\tilde{r}_{ij}$ approximation: First we consider a simple allocation scheme: allocate each cell to the agent with largest value of $\tilde{r}_{ij}$. The next proposition bounds the performance of this scheme in terms of a bound on the interference costs.

Proposition 1: Let $\gamma > 0$ be a constant so that

$$\sum_{j':(j,j')\in E} c^i_{jj'} \leq \gamma \tilde{r}_{ij} \quad \forall i \in A, j \in C. \tag{1}$$

Then allocating each cell to the agent with the largest $\tilde{r}_{ij}$ gives a $(1+\gamma)$-approximation to (P2).

Proof: Let $\hat{Z}$ be the total utility achieved by this algorithm and let $i^*(j)$ be the agent assigned cell j in this solution. Note that $\hat{Z} \geq Z|_{\tilde{c}=0} := \sum_{j\in C} \tilde{r}_{i^*(j),j}$, where $Z|_{\tilde{c}=0}$ can be viewed as the solution to a modified version of (P2) in which each of the $\tilde{c}^i_{jj'}$ terms are set to zero. Similarly, let $Z|_{\tilde{r}=0}$ be the solution to (P2) in which all of the $\tilde{r}_{ij}$ terms are set to zero and let $\hat{i}(j)$ be the agent assigned to cell j in the solution to this problem. Note that $Z|_{\tilde{c}=0} + Z|_{\tilde{r}=0} \geq Z_{opt}$. Furthermore, from (1), we have

$$Z|_{\tilde{r}=0} \;\; \leq \;\; \sum_{j\in C} \sum_{j':(j,j')\in E} c^{\hat{i}(j)}_{jj'} \leq \sum_{j \in C} \gamma \tilde{r}_{\hat{i}(j),j} \leq \gamma Z|_{\tilde{c}=0}.$$

Combining, we have $(1+\gamma)\hat{Z} \geq Z_{opt}$. ■

2) Max-r_{ij}: A related approximation is to assign each cell to the agent with the largest value of r_{ij}. By a similar proof this scheme has the following approximation bound.

Proposition 2: Let $\gamma' > 0$ be a constant so that $\sum_{j':(j,j')\in E} c^i_{jj'} \leq \gamma' r_{ij}$, for all $i \in A, j \in C$. Then allocating each cell to the agent with the largest r_{ij} gives a $1/(1+\gamma')$-approximation.

3) Edge coloring approximation: Find a proper edge coloring of $\tilde{G} = (C, \tilde{E})$, which divides $\tilde{E}$ into q disjoint sets $E_1, ..., E_q$, one for each color. Decompose (P2) into $q+1$ independent sub-problems on each of $E_1, ..., E_q$ and C. Each of the sub-problems then can be solved independently and we pick the best solution as the approximation.

Proposition 3: If a proper edge coloring of G can be found using q colors, then the preceding procedure gives a $(1+q)$-approximation.

This approximation factor is minimized by finding an edge coloring of G using the fewest colors, i.e., by setting q equal to the chromatic index χ of G. For a general graph determining χ is NP-complete, but it can be approximated to within 1 by the maximum degree plus one. Moreover, for certain graphs of interest such as regular lattices, χ is equal to the degree and a χ-edge coloring can be easily found.

4) GRA-approximation: Let $Z|_{\tilde{c}=0}$ and $Z|_{\tilde{r}=0}$ be defined as in the Max-$\tilde{r}_{ij}$ approximation. Since, $Z|_{\tilde{c}=0} + Z|_{\tilde{r}=0} \geq Z_{opt}$, it follows that either $Z|_{\tilde{c}=0} \geq 1/2 Z_{opt}$ or $Z|_{\tilde{r}=0} \geq 1/2 Z_{opt}$. As we have noted previously, finding $Z_{\tilde{c}=0}$ is easy. However, exactly finding $Z|_{\tilde{r}=0}$ is difficult in general. Indeed, by a similar argument as in the proof of Proposition 2 in [4] it can be shown that determining $Z|_{\tilde{r}=0}$ is NP-hard. Instead we consider approximating this by adapting the *Geometric Rounding Algorithm* (GRA) in [12]. This involves solving the natural LP relaxation to (P2) and then applying a randomized dependent rounding scheme to get an integer solution. The specific scheme in [12] is shown to give a constant factor approximation to the *Winner Determination Problem* (WDP) in a combinatorial auction with single-minded bidders. Specifically, the problem is to efficiently allocate a collection of distinct goods to a set of bidders, where each bidder only desires a specific subset of the goods. The approximation factor for the GRA scheme in [12] is equal to the maximum cardinality of the subset desired by any agent.

Finding $Z|_{\tilde{r}=0}$ can be viewed as a generalization of the WDP problem in which the goods are nodes to be allocated to each agent. Each agent i will only value pairs of goods (j, j') for which $c^i_{jj'} > 0$. However, in our case, agents are not single minded and may value multiple pairs, with an additive valuation across pairs. It can be seen that the results in [12] still apply with such a generalization, i.e., applying the GRA algorithm approximates $Z|_{\tilde{r}=0}$ with an approximation factor equal to the maximum cardinality of a subset desired by an agent (2 in our case). Using this we have the following bound.

Proposition 4: Taking the minimum of $Z|_{\tilde{c}=0}$ and the GRA approximation to $Z|_{\tilde{r}=0}$ is a 4-approximation.

A. Numerical Example

Here we present a numerical example that illustrates the performance of the preceding approximations. We assume a square lattice with nine cells and six agents ($|C| = 9$ and $|A| = 6$). The revenue depends on the distribution of the end users over the cells. For this example we assume that each agent wishes to serve a set of pre-assigned users, which are distributed over the region according to a spatial Poisson process with intensity μ. The area of each cell is normalized to one, and the locations of end-user groups are independent. We assume that an agent's revenue for a cell is proportional to the number of assigned end users within the cell. The interference cost is modeled as a loss in coverage area, due to an agent's inability to serve end users close to the cell boundary. Those users experience the largest interference from the neighboring cell when it is assigned to another agent. We therefore assume that only a fraction of the total cell area $1-\lambda$ can be used to serve end users. The interference cost is then proportional to the number of end users located within the corresponding boundary area.

Fig. 1 shows total revenue versus the interference area λ for the different approximation algorithms with spatial Poisson intensities $\mu = 5$ and $\mu = 20$ users per cell (assumed to be the same for all agents). These are compared with the upper bound on total revenue obtained from the relaxed linear program of (P1). Each point is an average over 200 realizations in which the locations of the end users are randomly generated. The agents' revenues and costs are then determined by the number of end users in the corresponding areas.

As expected, the figure shows that the approximation algorithms achieve close to the maximum revenue for small λ. The gap widens as both λ and the spatial intensity μ increase. We also observe that the GRA-approximation performs best

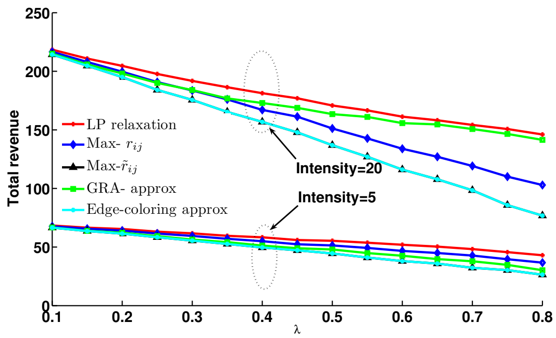

Fig. 1. Total revenue achieved by various approximation algorithms for a 3×3 lattice with six agents, and Poisson intensities of 5 and 20 users/cell.

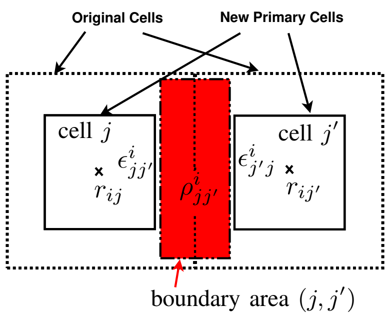

Fig. 2. Two adjacent cells showing the cell boundary area (shaded). The dashed lines represent the original cell boundaries and the solid lines represent new smaller cell boundaries.

for large λ and μ. However for smaller μ, the Max-r_{ij} approximation performs best. This algorithm assigns cells based on their total value assuming no interference, while the other algorithms make assignments based on either the total revenue from the cells' boundary areas or interference-free areas. For $\lambda > 0.5$, the probability that an agent has more revenue in the boundary areas than in the interference-free area is increasing with μ. Hence for large μ an algorithm that focuses on the boundary areas such as the GRA approximation performs better, while for smaller μ, the Max-r_{ij}, which accounts for the entire cell performs better.

IV. Markets with Secondary Cell-Edge Users

In practice interference primarily affects the users near cell boundaries, while users near the interior of a cell may receive little interference. In this section we present an alternate model in which the cell boundary areas are treated as separate assets from the cell interiors. For example, the cell boundary areas might be efficiently used by *secondary* agents that provide local service within those regions with lower Quality of Service. This model is partially motivated by the recent interest in secondary spectrum usage in which secondary users can make use of idle spectrum at a particular location provided that they do not interfere with primary users [7], [13].[8]

[8]This model has been adopted by the FCC for secondary use of TV white spaces [14].

The alternate model is illustrated in Fig. 2, which shows two adjacent cells. The locations of the cells are the same as previously defined, but the size of the cells are reduced so that the cells do not receive significant interference. This corresponds to the situation where the power mask for each cell is low enough such that the interference from neighboring cells can be ignored within the new smaller cells. As a result of the new smaller cells (with solid lines in Fig. 2), the area along the original cell boundary (shaded in Fig. 2) becomes available for secondary use. Here we assume that the spectrum assets in the cell boundary can be purchased by any agent. An agent acquiring a cell boundary must not cause significant interference to the neighboring primary users, where the specific amount of interference may be specified by the market-designer.

Next we describe the model in detail. As before, A denotes the set of agents and C denotes the set of primary spectrum assets. Again, these assets are related via an un-directed graph $G = (C, \tilde{E})$, where $\tilde{E}$ now represents the set of assets which share a boundary area, i.e., each boundary area is uniquely indexed by an edge in $\tilde{E}$. A boundary area $(j, j') \in \tilde{E}$ only experiences interference from the neighboring cells j and j'.

Agent i receives revenue r_{ij} when assigned cell j regardless of whether or not she is assigned the neighboring boundary areas. Let $\rho^i_{jj'}$ denote the revenue agent i receives from boundary area (j, j') in isolation. If the agent owns cell j and the neighboring boundary area (j, j'), then the agent receives an additional (complementary) revenue of $\epsilon^i_{jj'}$. This is again due to the possibility of mitigating interference by coordinating transmissions across the cell and boundary-area.

For the preceding model, the efficient allocation can be formulated as a total revenue maximization problem with objective

$$\sum_{i \in A} \Big(\sum_{j \in C} r_{ij} x_{ij} + \sum_{(j,j') \in \tilde{E}} (\rho^i_{jj'} y^i_{jj'} + \epsilon^i_{jj'} z^i_{jj'} + \epsilon^i_{j'j} z^i_{j'j}) \Big) \tag{P4}$$

The optimization is over the binary variables $\{x_{ij}, y^i_{jj'}, z^i_{jj'}\}$ where $x_{ij} = 1$ if agent i is assigned cell j, and is zero otherwise, $y^i_{jj'} = 1$ if the boundary-area between cells j and j' is assigned to agent i (zero otherwise), and $z^i_{jj'} = 1$ if agent i is assigned both cell j and the boundary (j, j') (zero otherwise). Note that $z^i_{jj'}$ is directional in the sense that $z^i_{jj'}$ and $z^i_{j'j}$ refer to different assets.

It can be shown that (P4) is equivalent to a special case of (P2) with a special structure in the underlying interference graph. Unfortunately, this structure does not make the problem more tractable. Indeed, using similar arguments as in [4] it can be shown that (P4) is still NP-hard. Moreover, from this equivalency it follows that the approximation algorithms in Sect. III apply to (P4) with similar approximation ratios. The structure in (P4) can also be used to identify a condition under which the problem can be solved by a simple greedy procedure. The precise statement follows.

Proposition 5: If for any boundary-area $(j, j') \in \tilde{E}$, there

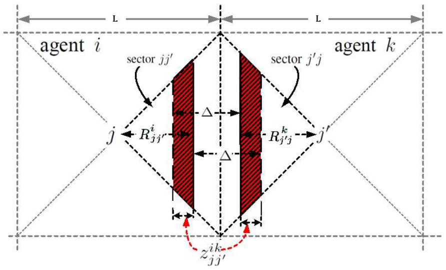

Fig. 3. A sectoried radii model for two adjacent square cells each with four sectors.

exists some agent i such that

$$\rho^i_{jj'} \geq \rho^{i'}_{jj'} + \epsilon^{i'}_{jj'} + \epsilon^{i'}_{j'j}$$

for all $i' \neq i$, then (P4) can be solved by a two-stage greedy algorithm which first assigns each boundary area to the agent with the largest value of $\rho^i_{jj'}$ and then assigns each cell to the agent with the largest revenue given the boundary assignment.

Proposition 5 can be proved simply by a backward induction argument, which we omit.

V. Markets with Flexible Cell Boundaries

In the models presented so far, a fixed interference cost is incurred by neighboring assets if they are not assigned to the same agent. The previous models are generic in the sense that this cost can arise from many scenarios. In this section, we consider a more specific model for mitigating interference, namely by adjusting the "radius" over which an agent can transmit in a given cell. For example, if agents serve users in each cell via downlink transmissions from a single access point this can be accomplished by adjusting the transmission power of the access point.[9] Moreover, we focus on a model in which the market mechanism assigns cells *along with associated cell radii* to the agents.

For a given radius, a cell incurs an interference cost if the interference boundary of a neighboring cell overlaps its transmission area. Here, "interference boundary" refers to the range of significant interference generated by a particular cell, which lies beyond the cell boundary.[10] We consider this model with and without sectorization. Compared to the original allocation mechanism in Sect. II, the additional flexibility of having variable cell radii will be shown to simplify the search for an efficient allocation.

Here we focus on the special case in which the underlying interference graph is a *square* lattice, and we again let $G = (C, \tilde{E})$ be the corresponding undirectional interference graph. Many of the following results can also be easily extended to other common topologies such as triangular or hexagonal lattices.

[9]In other cases, such as uplink transmissions, the model can be viewed as simply determining the radius within which users may transmit.

[10]This is similar to the interference footprint in the standard protocol model from [15].

A. Model with Sectorization

Fig. 3 illustrates the radii model for two adjacent square cells. The length of a cell edge is L, which is also the distance between the centers of neighboring cells. Each cell in the figure has 90-degree sectors. For example this could correspond to the case where an access point in the center of the cell uses directional antennas, so that the transmit power/cell radius for each sector can be independently determined. As shown, we assume that each sector experiences interference from only the closest sector in the neighboring cell. We can identify each edge $(j, j') \in \tilde{E}$ with a pair of interfering sectors; we will abuse notation and denote the corresponding sector in cell j (or j') by jj' (or $j'j$).

Let $R^i_{jj'} \in [0, L/2]$ be the *radius* of agent i in sector jj', which is the minimal distance from the cell center to its boundary over which agent i can serve customers in the absence of interference. Let w_{ij} be the revenue per unit area of agent i in cell j. For example, w_{ij} can be related to the density of customers agent i serves in cell j. Agent i's revenue from sector jj' is then taken to be the revenue density times the sector area $w_{ij} R^{i2}_{jj'}$ in the absence of interference from the opposite sector $j'j$.

Interference costs are incurred when the interference boundaries of two neighboring cell sectors overlap. Specifically, $R^i_{jj'} + R^k_{j'j} \geq L - \Delta$, where Δ is the width of an interference guard zone. That is, interference from cell j in sector $(j'j)$ can be ignored beyond distance $R^i_{jj'} + \Delta$ from the center of cell j. Hence if $R^i_{jj'} + R^k_{j'j} \geq L - \Delta$, then agent i's coverage in sector (jj') will create interference to agent k in sector $(j'j)$, and vice versa (agent k will interfere with i). Let $z^{ik}_{jj'} = \max\{R^i_{jj'} + R^k_{j'j} - (L - \Delta), 0\}$ denote the amount of overlap of interference boundaries in such neighboring sectors. Fig. 3 shows a scenario where $z^{ik}_{jj'} > 0$. The shaded areas are where the interference occurs. We assume that interference externalities (i.e., loss in revenue due to interference) are proportional to the area of the shaded regions since users within those regions may not receive adequate service. Hence the revenue of agent i in sector jj' is $w_{ij}(R^i_{jj'} - z^{ik}_{jj'})^2$.

No explicit coordination is assumed here to avoid interference between neighboring sectors assigned to the same agent. Namely, $z^{ii}_{jj'}$ is not necessarily zero. This is because the market is able to manage interference among the sectors assigned to the same agent by optimizing over the corresponding radii of the sectors.

The efficient allocation problem for this model can be written as the following *mixed-integer quadratic program* (MIQP) :

$$\max_{\mathbf{x},\mathbf{R},\mathbf{z}} \sum_{i \in A} \sum_{j \in C} \sum_{j':(j,j') \in E} w_{ij} (R^i_{jj'} - \sum_{k \in A} z^{ik}_{jj'})^2 \qquad \text{(P5)}$$

$$\text{s.t. } x_{ij}(\frac{L}{2} - \Delta) \leq R^i_{jj'} \leq x_{ij} \frac{L}{2}, \ \forall i \in A, \ (j, j') \in E$$

$$0 \leq R^i_{jj'} + R^k_{j'j} - z^{ik}_{jj'} \leq L - \Delta, \ \forall i, k \in A, (j, j') \in \tilde{E}$$

$$\sum_i x_{ij} \leq 1, \ \forall j \in C$$

$$z_{jj'}^{ik} \geq 0\,, \forall (j,j') \in \tilde{E},\ x_{ij} \in \{0,1\}\ \forall i,k \in A,\ j \in C$$

This is clearly a simplified model that we have chosen to highlight the potential advantages of having a market determine both cell assignments and the assignment of power/radii. Before discussing the solution to (P5), we briefly comment on a few of these simplifications. The assumption that cells are located on a regular lattice is one simplification; this could be relaxed for example by introducing different distances $L_{jj'}$ for different pairs of cells. Another simplifying assumption is the model for interference costs; one could use a more sophisticated physical layer model to capture these effects and moreover the cost of such interference could vary among providers who use different technologies and/or have different QoS requirements. Finally, when determining the revenue of a provider, we do not account for any constraints on the number of users a provider can serve within a cell. This is reasonable if spectrum is relatively abundant; if this is not the case then adding such constraints appears to make the problem more difficult.

Problem (P5) can be solved via the following two-step procedure: first, determine an assignment of cells to each agent and second, determine the radii of each sector for the assigned cells. The following lemma shows that the first step in this procedure can be solved independently of the second.

Lemma 1: If $\{x_{ij}^*\}$ is a solution to (P5), then $x_{ij}^* = 1$ if and only if $i = \arg\max_i w_{ij}$ for each cell $j \in C$.

Proof: Let $\{\tilde{x}_{ij}, \tilde{R}^i_{jj'}, \tilde{z}^{ik}_{jj'}\}$ be an optimal solution to (P5) and suppose that the lemma is not true for some cell j. Let $\tilde{i}$ be the user currently assigned cell j and let $i^* = \arg\max_i w_{ij}$. Then consider a new solution where cell j is assigned to agent i^* with the same radii for each sector and the same choice of the corresponding $z^{ik}_{jj'}$ as assigned to user $\tilde{i}$ in the original solution. All other variables are unchanged. It can be seen that this must still be a feasible solution with the same area served in each cell. Moreover, the revenue density in cell j will increase, resulting in a larger total revenue. Hence, the original solution cannot be optimal, proving the lemma. ■

Given an optimal cell assignment as in Lemma 1, we next consider optimizing the cell radii. This is given by the following quadratic program (QP):

$$\max_{\mathbf{R},\mathbf{z}} \sum_{(j,j') \in \tilde{E}} w_j (R_{jj'} - z_{jj'})^2 + w_{j'} (R_{j'j} - z_{jj'})^2 \qquad \text{(P6)}$$
$$\text{s.t.}\ \ 0 \leq R_{jj'} + R_{j'j} - z_{jj'} \leq L - \Delta,\ \forall\, (j,j') \in \tilde{E}$$
$$\frac{L}{2} - \Delta \leq R_{jj'} \leq \frac{L}{2},\ \forall\, (j,j') \in E$$
$$z_{jj'} \geq 0,\ \forall\, (j,j') \in \tilde{E}$$

where we have dropped the agent indicies, since the agent assigned to each cell is given. The objective of this QP is convex and so it cannot be solved directly by using first order conditions. However, its extreme points have the following useful property:

Lemma 2: An optimal solution to (P6) must satisfy $R_j^* \in \{L/2 - \Delta, (L-\Delta)/2, L/2\}$.

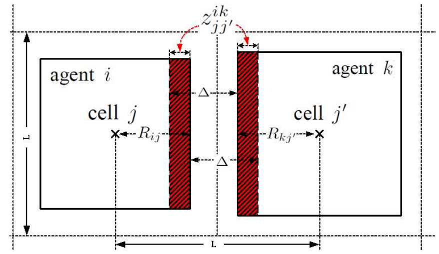

Fig. 4. Illustration of the omnidirectional radii model for a square lattice.

The proof of this uses the same technique in the proof of Proposition 2.1 in [16] and so we omit it here.

Since each sector jj' only interferes with the neighboring sector $j'j$, it can be seen that (P6) can be separated into a collection of sub-problems, one for each pair of neighboring sectors $(j,j') \in E$. Furthermore, the subproblem for $(j,j') \in E$ only involve the variables $R_{jj'}$, $R_{j'j}$ and $z_{jj'}$, which from Lemma 2 can take on only a finite number of values each. Hence, we can solve (P6) and thus (P5) in polynomial-time.

The key difference between the current model and the original model in Sect. II is due to the flexibility in having the market assign the radii. Indeed if the radii are not determined by the market, then it can be shown that this model is equivalent to a special case of the model in Sect. II which is still NP-hard. Specifically, suppose that an agent always uses the maximum radius $L/2$ if she is assigned sector jj' but not sector $j'j$, so that $z^{ik}_{jj'} = \Delta$. If agent i is assigned both sectors jj' and $j'j$, she can optimize the radii of the two sectors herself by solving the corresponding sub-problem in (P6). We can then map this to the problem in Sect. II by letting r_{ij} be the revenue obtained from cell j using the optimal radii $\{R^{i*}_{jj'}\}_{j':(j,j') \in E}$ and letting the interference cost $c^i_{jj'}$ be the cost to agent i from using $L/2$ instead of the optimal radius $R^{i*}_{jj'}$, i.e., $c^i_{jj'} = w_{ij}(R^{i*}_{jj'})^2 - w_{ij}(L/2 - \Delta)^2$.

B. Omnidirectional Model

In this section we consider a variation of the radii model in which cells are not sectorized so that each cell has the same radii in each direction. For example, this could model a system in which agents transmit from an access point in the center of a cell using an omnidirectional antenna. With sectorization, the optimization of cell radii for a given assignment is decomposed into a separate problem for each pair of interfering sectors. In an omnidirectional model this is no longer the case and the optimization of cell radii becomes coupled across multiple cells. Nevertheless, we will show that a linearized version of this problem can still be efficiently solved.

The model we consider here is the same as in the previous section except a single cell radius R_{ij} is used for each sector of a cell j assigned to agent i, where again this denotes the minimal distance from the cell center to its boundary. This means that the resulting cells are always squares as shown in Fig. 4. Agent i's revenue when it is assigned a cell j with radius R_{ij} is then $4w_{ij}R_{ij}^2$ minus interference costs resulting from any overlap with the interference footprint of neighboring cells (shown as a shaded area in Fig. 4). In this section, we assume that the interference footprints are also squares, i.e., interference from a neighbor is not limited to a single 90 degree sector. The revenue agent i receives from cell j is then given by

$$w_{ij}\left(2R_{ij}-\sum_{k\in A}(z^{ik}_{jj^n}+z^{ik}_{jj^s})\right)\left(2R_{ij}-\sum_{k\in A}(z^{ik}_{jj^w}+z^{ik}_{jj^e})\right) \tag{2}$$

where $z^{ik}_{jj'}$ again denotes the amount of overlap of a neighboring cell's interference area and j^n, j^s, j^w, and j^e denote the cells to the north, south, east and west of j (with respect to an arbitrary choice of north). The efficient allocation is then given by the following MIQP:

$$\begin{aligned}\max_{\mathbf{x},\mathbf{R},\mathbf{z}} &\sum_{i\in A}\sum_{j\in C} w_{ij}\left(2R_{ij}-\sum_{k\in A}(z^{ik}_{jj^n}+z^{ik}_{jj^s})\right) \qquad \text{(P7)}\\ &\times\left(2R_{ij}-\sum_{k\in A}(z^{ik}_{jj^w}+z^{ik}_{jj^e})\right)\\ \text{s.t. } & 0\le R_{ij}+R_{kj'}-z^{ik}_{jj'}\le L-\Delta,\ \forall i,k\in A,(j,j')\in\tilde{E}\\ & x_{ij}(\frac{L}{2}-\Delta)\le R_{ij}\le x_{ij}\frac{L}{2},\ \forall i\in A,\ j\in C\\ & \sum_i x_{ij}\le 1,\ \forall j\in C\\ & z^{ik}_{jj'}\ge 0,\ \forall (j,j')\in\tilde{E},\ x_{ij}\in\{0,1\},\ \forall i,k\in A,\ j\in C\end{aligned}$$

Lemmas 1 and 2 can be generalized to this problem. However, given an assignment of cells to users, the resulting QP for determining the cell radii is coupled across the cells due to the omnidirectional radii. Moreover, the objective is neither concave or convex, making this difficult to solve for a large number of cells. However, note that after making the assignment of cells to users the number of remaining variables is much smaller; there will be no more than $3|C|$ variables while before making an assignment the number of variables are on the order of $2|A|^2|C|+|A||C|$; hence for a moderate number of cells, it is feasible to use a commercial solver to determine the optimal radii.[11] Alternatively, we next consider a linearized version of this problem which yields a more tractable solution.

Note that since $R_{ij}\le L/2$, no interference costs will be incurred if an agent uses a radius $L/2-\Delta$. Thus, the revenue that an agent gains from a cell can be represented as the sum of the revenue from a square with radius $L/2-\Delta$ and the remaining area, which may incur an interference cost. Specifically, by replacing R_{ij} in (2) with $(L/2-\Delta)+(R_{ij}-(L/2-\Delta))$ and simplifying the resulting expression we have:

$$w_{ij}\left(2R_{ij}-\sum_{k\in A}(z^{ik}_{jj^n}+z^{ik}_{jj^s})\right)\left(2R_{ij}-\sum_{k\in A}(z^{ik}_{jj^w}+z^{ik}_{jj^e})\right)$$

[11]Note that by using Lemma 2 this can be formulated as an integer QP.

$$\begin{aligned}&=8\left(\frac{L}{2}-\Delta\right)\left[w_{ij}R_{ij}-\sum_{j':(j,j')\in E}\frac{1}{4}w_{ij}\sum_{k\in A}z^{ik}_{jj'}\right]\\ &-4w_{ij}\left(\frac{L}{2}-\Delta\right)^2+O(\Delta^2),\end{aligned}$$

where here we have used the fact that $R_{ij}\le L/2$ and $z^{ik}_{jj'}\le\Delta$ to get the $O(\Delta^2)$ bound on the missing terms.

If we drop the $O(\Delta^2)$ terms then the remaining terms are linear in the optimization variables. Furthermore, without loss of generality we can drop the constant term $4w_{ij}(\frac{L}{2}-\Delta)^2$, and the scaling term of $8\left(\frac{L}{2}-\Delta\right)$ (which is always non-negative). This gives the following linear expression for the revenue in cell j:

$$w_{ij}R_{ij}-\sum_{j':(j,j')\in E}\sum_{k\in A}\alpha w_{ij}z^{ik}_{jj'} \tag{3}$$

where for a square lattice $\alpha=\frac{1}{4}$. A similar objective can be derived for triangular or hexagonal lattices, where α is $1/3$ or $1/6$, respectively. Using this as an agent's revenue, the efficient allocation is given by

$$\begin{aligned}\max_{\mathbf{x},\mathbf{R},\mathbf{z}} &\sum_{i\in A}\sum_{j\in C}\left(w_{ij}R_{ij}-\sum_{j':(j,j')\in E}\sum_{k\in A}\alpha w_{ij}z^{ik}_{jj'}\right) \qquad \text{(P8)}\\ \text{s.t. } & (\mathbf{x},\mathbf{R},\mathbf{z})\in\mathcal{P}\end{aligned}$$

where $\mathcal{P}$ denotes the same constraint set as in (P7).

Lemma 1 can be extended to (P8) and so this problem can again be solved by first allocating each cell to the agent with the largest w_{ij} and then optimizing the radii given this assignment. In (P8), the second step is a linear program, which can be efficiently solved.

If we take the linear formulation in (P8) as the objective, then with omnidirectional radii we can again find the efficient solution provided that the market determines the optimal radii. Of course, if the true valuation is given by the objective in (P7), then solving this linear version will lead to a loss in revenue. The ratio of the loss in revenue from this approximation and the optimal revenue from (P7) can be shown to be no greater than $8\Delta/L$ for a square lattice, suggesting that this approximation is the most reasonable for small Δ. Numerical results in the next section, show that the loss may be small even for large values of Δ.

C. Numerical Examples

In this section we present some numerical results to compare the revenue achieved by the original model and the radii models both with and without sectorization. We focus on a square lattice with 4×4 cells and six agents. The cell size is normalized to 1, so that $L=1$, and for the radii model the revenue density (w_{ij}) for each agent in each cell is randomly generated following a Poisson distribution with intensity $\mu=50$. As in Sect. III-A, we let $1-\lambda$ denote the fraction of each cell which is always interference free. For the radii models, this is equivalent to choosing $\Delta=(1-\sqrt{1-\lambda})/2$. We choose the revenues and costs for the original model by using the mapping described in Sect. V-A between the sectorized model and the original model. For the original model, we exactly solve the integer program. For the omnidirectional model, we solve the linear approximation to determine the cell assignment and radii used, but then plot the corresponding revenue using the original quadratic formulation. We also directly solve the original quadratic formulation numerically. As a benchmark, we also show results for a model with spatial guard bands, i.e., one in which each agent is required to always use a radius of $L/2 - \Delta$ so that no two cells interfere. In this case the efficient allocation is simply to assign each cell j to the agent with the largest value of w_{ij}. Figure 5 shows the total

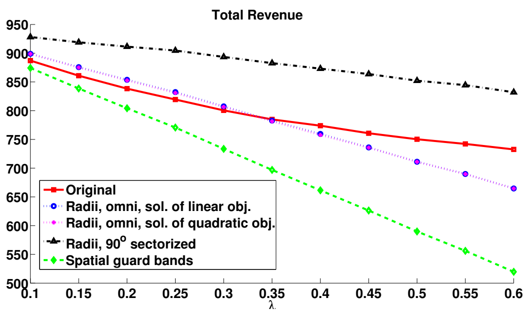

Fig. 5. Total revenue in different market models.

revenue achieved for each of these models versus λ. Each point is averaged over 200 realizations of the w_{ij}'s, with the same set of realizations used for each model. As expected, the model with fixed spatial guard bands achieves the lowest revenue, which goes to zero as λ approaches 1 since no revenue is obtained from users in the guard zones. Even for moderate values of λ all of the other approaches achieve significantly higher revenue, demonstrating the potential benefit from allowing the spectrum market to manage interference. The highest revenue is achieved by the radii model with sectorization which is expected as this offers the most flexibility in assigning resources. Both curves for the omnidirectional radii model are indistinguishable for the entire range of λ, suggesting that at least for this scenario, the linear model is a good approximation for the quadratic model. For small enough λ (equivalently small Δ), the omnidirectional radii model outperforms the original model. However, for relatively large λ (large Δ), the revenue from the omnidirectional radii models is less than that from the original model. Note that while the omnidirectional model has the flexibility of optimizing cell radii across agents, the costs for the original model are based on allowing agents to adapt cell radii on a sector basis when they own neighboring cells. Hence, it is not clear that one of these schemes should always dominate the other. For large λ the revenue from "boundary regions" is greater, and apparently in that case the original model has better performance.

VI. Conclusions

We have examined several simple models of spectrum markets with interference complementarities to demonstrate how different assumptions on the market structure can impact both the computation of an efficient outcome and the resulting revenue. We began with a basic linear model in which the market specified only the assignment of cells to users. Since determining an efficient assignment of cells to agents was NP-hard, we focused on several constant factor approximations, which had good performance in numerical examples. Next we considered a market where guard zones between primary cells were allocated for secondary use. This did not improve the complexity of the finding an efficient allocation, but does provide structure that can be exploited in several cases. Next, we considered models in which in addition to the assignment of cells the market also determined a cell radii, which led to simpler allocation problems as well as higher total revenue.

Future research directions include developing more refined models for interference costs and studying their effect on the resulting markets, studying strategic behaviors of agents in such markets and developing mechanisms to implement the markets.

References

[1] M. M. Bykowsky, M. A. Olson, and W. W. Sharkey, "A market-based approach to establishing licensing rules: Licensed versus unlicensed use of spectrum," *FCC OSP Working Paper Series No.43*, Feb. 2008.

[2] L. Doyle and T. Forde, "Towards a fluid spectrum market for exclusive usage rights," *IEEE DySPAN 2007*, pp. 620–632, 17-20 April 2007.

[3] FCC, "Second report and order: Promoting efficient use of spectrum through elimination of barriers to the development of secondary markets," FCC Report 04-167, Sept. 2004.

[4] H. Zhou, R. Berry, M. Honig, and R. Vohra, "Complementarities in spectrum markets," in *Proc. 47th Annual Allerton Conference*, Sept. 2009.

[5] X. Zhou and H. Zheng, "Trust: A general framework for truthful double spectrum auctions," in *Proc. IEEE INFOCOM*, pp. 999-1007, 2009.

[6] J. Jia, Q. Zhang, Q. Zhang, and M. Liu, "Revenue generation for truthful spectrum auction in dynamic spectrum access," in *Proc. ACM MobiHoc*, pp. 3–12, 2009.

[7] G. S. Kasbekar and S. Sarkar, "Spectrum auction framework for access allocation in cognitive radio networks," in *Proc. ACM MobiHoc '09*, pp. 13–22, 2009.

[8] O. Ileri, D. Samardzija, and N. Mandayam, "Demand responsive pricing and competitive spectrum allocation via a spectrum server," in *Proc. of IEEE DySPAN 2005*, pp. 194–202, Nov. 2005.

[9] Y. Xing, R. Chandramouli, and C. Cordeiro, "Price dynamics in competitive agile spectrum access markets," *IEEE Journal on Selected Areas in Communications*, vol. 25, no. 3, pp. 613–621, April 2007.

[10] D. Niyato and E. Hossain, "Market-equilibrium, competitive, and cooperative pricing for spectrum sharing in cognitive radio networks: Analysis and comparison," *IEEE Trans. on Wireless Communications*, vol. 7, no. 11, pp. 4273 –4283, Nov. 2008.

[11] R. Berry, M. L. Honig, and R. Vohra, "Spectrum markets: Motivation, challenges and implications," *IEEE Communication Magazine*, vol. 48, no. 11, Nov. 2010.

[12] D. Ge, S. He, Z. Wang, Y. Ye, and S. Zhang, "Geometric rounding: A dependent rounding scheme for allocation problems," April 2008. [Online]. Available: http://www.optimization-online.org/DB_HTML/2008/04/1949.html

[13] N. Chang and M. Liu, "Optimal competitive algorithms for opportunistic spectrum access," *IEEE Journal on Selected Areas in Communications*, vol. 26, no. 7, pp. 1183–1192, Sept. 2008.

[14] FCC, "In the matter of unlicensed operation in the TV broadcast bands: Second report and order and memorandum option and order," Tech. Report, Nov. 2008.

[15] P. Gupta and P.R. Kumar, "The capacity of wireless networks," *IEEE Trans. Info. Theory*, vol. 46, no. 2, pp. 388-404, March 2000.

[16] G. L. Nemhauser and L. E. Trotter, "Properties of vertex packing and independence system polyhedra," *Mathematical Programming*, vol. Volume 6, Number 1, pp. 48–61, December, 1974.