Distributed Power Allocation and Scheduling for Parallel Channel Wireless Networks

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Abstract

In this paper, we develop distributed approaches for power allocation and scheduling in wireless access networks. We consider a model where users communicate over a set of parallel multi-access fading channels, as in an OFDM or multi-carrier system. At each time, each user must decide which channels to transmit on and how to allocate its power over these channels. We give distributed power allocation and scheduling policies where each user's actions depend only on knowledge of their own channel gains. We characterize an optimal policy which maximizes the system throughput and also give a simpler sub-optimal policy which is shown to have the optimal scaling behavior in several asymptotic regimes.

I. Introduction

It is well established that dynamically allocating transmission rate and power are beneficial techniques to improve performance of wireless networks. In this paper we consider these approaches for the uplink in a wireless network, modeled as a fading multiple access channel. For such channels, power allocation and scheduling have received much attention. For example, [1]–[4] consider these problems in the context of the information theoretic capacity region of a multi-access fading channel under various assumptions. In other work, such as [5], [9], adaptive bit and power allocation are studied in the context of an OFDM system. In these cases, optimally allocating resources requires a centralized controller with knowledge of every user's channel state. Because of the required overhead and delays involved, this may not be feasible in a fast-fading environment or a system with a large number of users. Here, we instead consider distributed approaches, where each transmitter allocates its transmission rate and power based only on knowledge of its own channel conditions. This can be obtained for example via a pilot signal broadcast by the receiver in a time-division duplex system. Previously, in [6], [7], we considered distributed scheduling when all users communicate over a single flat fading channel. In this paper, we consider the case where each user may transmit over multiple "parallel" channels. For example, each channel may model a subcarrier or groups of subcarriers in an OFDM system such as the IEEE 802.16. In this case, a user must also decide how to allocate its transmission power across the available channels.

We first formulate a distributed power allocation and scheduling problem with a finite number of users and give the optimal solution to this problem. We next give a simplified allocation scheme and analyze the performance of both schemes in three asymptotic regimes: (*i*) the number of users increases with a fixed number of channels, (*ii*) the number of channels increase with a fixed number of users, and (*iii*) both the number of channels and the number of users increase with fixed ratio. In each case, we characterize the asymptotic growth rate. The simplified distributed approach is shown to achieve the same order of growth as an optimal centralized approach with a different asymptotic ratio in each regime. We also compare the performance of several approaches that require a small amount of additional coordination among the users. Finally, some simulation results are given.

II. Optimal Distributed Power Allocation for Parallel Channels

We consider a model of the uplink in a wireless network with N users sharing K parallel channels and all transmitting to a common receiver. Each channel between each user and the receiver is modeled as a time-slotted, block-fading channel; if only the nth user transmits on channel k in a given time-slot, the received signal, $y_{nk}(t)$

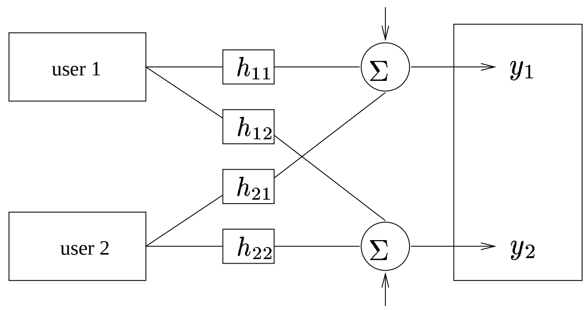

Fig. 1. Two users with $K = 2$ parallel channels.

is given by

$$y_{nk}(t) = \sqrt{H_{nk}} x_{nk}(t) + z(t), \tag{1}$$

where $x_{nk}(t)$ is the transmitted signal, H_{nk} is the fading channel gain of the kth channel for user n, and $z(t)$ is additive white Gaussian noise. The channel gain is assumed to be fixed during each time-slot and to randomly vary between time-slots. Here, $\{H_{n,k}\}$ are i.i.d. across both the users and channels with a continuous probability density $f_H(h)$ on $[0, \infty)$ [1]. Let $F_H(h) = \int_h^\infty f_H(h)\,dh$ denote the channel gain's complimentary distribution function.

At the start of each slot, user n has knowledge of $H_{n,1}, ..., H_{n,K}$, but no knowledge of the channel gains for any other users. For convenience, we drop the user subscript and denote $\mathbf{H} = (H_1, ..., H_k)$. Assume each user allocates power $P_k(\mathbf{H})$ to channel k, and let $\mathbf{P}(\mathbf{H}) = (P_1(\mathbf{H}), P_2(\mathbf{H}), ..., P_k(\mathbf{H}))$. A power constraint P_m constrains the total power allocated by each user across all K channels during any time-slot, *i.e.* $\sum_i P_i(\mathbf{H}) \leq P_m$. We assume no cooperation exists among users. In particular, we assume that all users employ the same power allocation and transmission scheme; *i.e.* they can not cooperate in selecting these allocations. Similar to [6], we consider an Aloha type of approach, where each user randomly transmits over each channel. If more than one user transmits on a given channel, a collision occurs and no packets are received; however, a packet sent over another channel without a collision will still be received. In other words, if a user simultaneously transmits on multiple channels, then the information sent over each channel is independently encoded, so that a packet sent on one channel may be decoded even if a collision occurs on another. This random access technique can also be used as a method of reservation in a practical system.

When only one user transmits on channel k, we model the user's transmission rate on this channel by

$$C(H_k P_k(\mathbf{H})) = W \log(1 + \frac{H_k P_k(\mathbf{H})}{N_0 W}),$$

where this indicates the Shannon capacity for a Gaussian noise channel with noise power $N_o W$ and bandwidth W. This is a reasonable model for systems using sophisticated coding techniques such as Turbo codes. To begin, consider the case where only one user must allocate the power over $K = 2$ channels. Let $N_0 W = 1$ to simplify notation. In this case the power allocation that maximizes a users throughput is the well-known "water-filling" allocation, $P_k(\mathbf{h}) = (\lambda - \frac{\mathbf{1}}{\mathbf{h_k}})^+$, where λ is chosen so that $\sum_k P_k(\mathbf{h}) = \mathbf{P_m}$ (see e.g. [1]).

When there are multiple users, if more than one user transmits on a channel a collision results and no data is received. Still assuming $K = 2$, consider the case where there are $N > 1$ users, and each user transmits on each channel with a certain probability p. Specifically, for each channel k, each user chooses a subset $\mathcal{H}_k$ of the possible realizations of $\mathbf{H}$ with $\Pr(\mathbf{H} \in \mathcal{H}_k) = p$ and only transmits on channel k when $\mathbf{H} \in \mathcal{H}_\mathbf{k}$. To maximize the total throughput, each user will choose channel states in each set $\mathcal{H}_k$ which can achieve higher transmission rates. The difficulty here is that if a state $\mathbf{H}$ is in both $\mathcal{H}_1$ and $\mathcal{H}_2$, the user must allocate power across both channels, while if $\mathbf{H}$ is in only one of these sets, the user can use all the available power on the corresponding channel. Since each channel is i.i.d., it is reasonable to require that each user transmits with the same probability p in each slot and each channel. The probability of some user successfully transmitting on one channel is then $Np(1-p)^{N-1}$. When successful, the transmission rate for channel k is still $C(H_k P_k(\mathbf{H}))$ for $k = 1, 2$. For a given power allocation P_k, let

$$R_k(P_k) = \mathbb{E}_\mathbf{H}\left(C(H_k P_k(\mathbf{H}))\middle| P_k(\mathbf{H}) \neq 0\right)$$

denote the expected transmission rate on the channel k conditioned on a user transmitting on that channel. We now specify the following *distributed optimal throughput problem*:

$$\begin{aligned} &\max_{P_1(\mathbf{H}), P_2(\mathbf{H}), p} Np(1-p)^{N-1}(R_1(P_1) + R_2(P_2)) \\ \text{s.t.} \quad &P_1(\mathbf{h}) + P_2(\mathbf{h}) \leq P_m, \ \forall \mathbf{h} \\ &Pr\{P_k(\mathbf{H}) \neq 0\} = p, \ k = 1, 2. \end{aligned} \tag{2}$$

The objective in (2) is the total average throughput for all N users over both channels. This is optimized over the transmission probability p and the power allocation $(P_1(\mathbf{H}), P_2(\mathbf{H}))$, which is used by each user. An example of the optimal allocation is shown in Fig. 2, where the double crossed area represents when users transmit on both channels, and the single crossed area is when they transmit on only one channel.

[1] In an OFDM system different sub-carriers will typically experience correlated fading. However, if each channel is a large enough group of sub-carriers, then this independence assumption is reasonable.

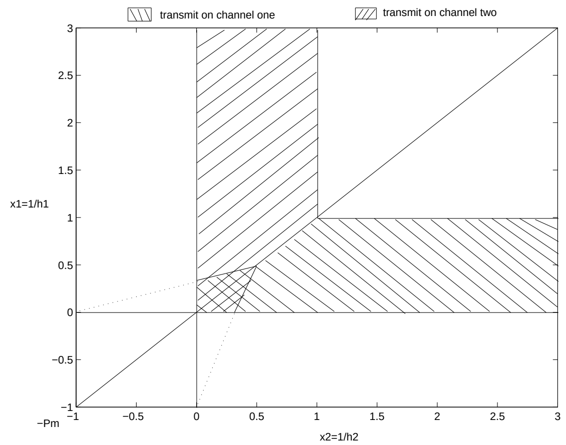

Fig. 2. Optimal power allocation for $K = 2$ parallel channels. The double crossed area indicates when the user transmits on both channels. The single crossed area indicates transmission on only one channel.

This can be naturally generalized to a model with K channels as follows:

$$\begin{aligned} &\max_{\mathbf{P}(\mathbf{H}),p} Np(1-p)^{N-1}\sum_{k=1}^{K} R_k(P_k) \\ \text{s.t. } &\sum_{k} P_k(\mathbf{h}) \leq P_m, \ \forall \mathbf{h} \\ &Pr\{P_k(\mathbf{H}) \neq 0\} = p, k = 1, ..., K. \end{aligned} \tag{3}$$

To solve (3), we first consider a different but related problem. For a given channel realization $\mathbf{H}$, let $(h_{(1)}, h_{(2)}, ..., h_{(K)})$ denote the ordered channel gains from the largest to the smallest. For $k = 1, ..., K$, let $R^k_{(i)}$ denote the rate on channel i when using the optimal water-filling power allocation over only the k best channels given total power P_m. For a given constant R_{th}, consider the following problem:

$$\begin{aligned} &\max_{k=1,\dots,K} k \\ \text{s.t. } &\sum_{i=1}^{k} R^k_{(i)} - \sum_{i=1}^{k-1} R^{k-1}_{(i)} \geq R_{th} \end{aligned} \tag{4}$$

If this problem has no feasible solution, we define the solution to be $k = 0$. When $k = 1$, the constraint in (4) is $R^1_{(1)} \geq R_{th}$, which means that the rate when only transmitting on the best channel should be greater than R_{th}. For $k = 2$, the constraint in (4) becomes $R^2_{(1)} + R^2_{(2)} - R^1_{(1)} \geq R_{th}$, which means that the rate gained by using the best two channels over only using the best channel is greater than R_{th}. In general, the objective of (4) is to find the maximal number of channels k, such that the gain of the sum rate from transmitting on the k best channels instead of only the $k-1$ best channels is no less than R_{th}. First, we have the following lemma, which says as we transmit on more channels, the change in the sum rate is non-increasing.

Lemma 1: $\sum_{i=1}^{k} R^k_{(i)} - \sum_{i=1}^{k-1} R^{k-1}_{(i)}$ is non-decreasing as k increases.

Proof: Consider the optimal (water-filling) power allocation over the following $2(k-1)$ channels $h_{(1)}, h_{(1)}, h_{(2)}, h_{(2)}, ..., h_{(k-1)}, h_{(k-1)}$, with total power $2P_m$. The resulting sum rate will be $2\sum_{i=1}^{k-1} R^{k-1}_{(i)}$. Next, consider $\sum_{i=1}^{k} R^k_{(i)} + \sum_{i=1}^{k-2} R^{k-2}_{(i)}$, this rate can be achieved by some power allocation over $2(k-1)$ channels with channel gains $h_{(1)}, h_{(1)}, ..., h_{(k-2)}, h_{(k-2)}, h_{(k-1)}, h_{(k)}$ and the same power constraint $2P_m$. Since $h_{(k)} < h_{(k-1)}$, and we are using the optimal power allocation for the first case, it can be seen that

$$2\sum_{i=1}^{k-1} R^{k-1}_{(i)} \geq \sum_{i=1}^{k} R^k_{(i)} + \sum_{i=1}^{k-2} R^{k-2}_{(i)}.$$

Therefore, rearranging terms we have

$$\sum_{i=1}^{k-1} R^{k-1}_{(i)} - \sum_{i=1}^{k-2} R^{k-2}_{(i)} \geq \sum_{i=1}^{k} R^k_{(i)} - \sum_{i=1}^{k-1} R^{k-1}_{(i)}.$$

■

Proposition 1: There exists a constant $R_{th} > 0$ such that the power allocation $\mathbf{P}(\mathbf{H})$ corresponding to the solution to (4), for each state $\mathbf{H}$ is also the optimal solution to (3).

Proof: The problem in (3) can be solved by the following two steps: first, given any p, find the optimal $\mathbf{P}(\mathbf{H})$, and second, find the optimal p for $\mathbf{P}(\mathbf{H})$ given in the first step. Given any R_{th}, the solution to (4) gives a unique p and vice versa. Also, note that R_{th} is monotonically decreasing as p increases. We prove that (4) gives the solution to the first step of solving (3), i.e. for a given p the optimal $P(H)$ will be given by the solution to (4) for some R_{th}. It follows that the optimal solution to (3) is then given by finding the optimal p or equivalently the optimal R_{th}.

To begin, we have two simple observations about the optimal power allocation: First, given a channel realization $(h_{(1)}, h_{(2)}, ..., h_{(K)})$, if the ith best channel (i) is allocated positive power, then the channels $h_{(1)}, .., h_{(i-1)}$ whose gains are better than $h(i)$ should also be allocated positive power; second, for any chosen channel states $h_{(1)}, .., h_{(k)}$, in order to maximize the total transmission rate, the water-filling power allocation should be used. Also note that from Lemma 1, if $\sum_{i=1}^{k} R^k_{(i)} - \sum_{i=1}^{k-1} R^{k-1}_{(i)} < R_{th}$, then $\sum_{i=1}^{m} R^m_{(i)} - \sum_{i=1}^{m-1} R^{m-1}_{(i)} < R_{th}$ for all $m > k$.

Now we complete the proof by contradiction. Let $\mathbb{H}$ be the K dimensional space of possible channel state vectors $\mathbf{h} = \mathbf{h_1}, ...\mathbf{h_K}$. Also, let $S_k = \sum_{i=1}^{k} R_{(i)}^{k}$ and $S_{k-1} = \sum_{i=1}^{k-1} R_{(i)}^{k-1}$. Assume there is an optimal power allocation for a given p is not a solution to (4). Let R_{th} be the threshold such that the solution to (4) using R_{th} results in a transmission probability p. Since the optimal power allocation does not correspond to this, for some channel l, there must exist a set of states $\mathcal{N} \in \mathbb{H}$ such that when l is the kth best channel and $S_k < S_{k-1} + R_{th}$, l is allocated positive power. Likewise, to ensure that $Pr\{P_k(\mathbf{H}) \neq 0\} = p$, there must be another set of states $\mathcal{B} \in \mathbb{H}$ (with the same probability) such that when l is the kth best state and $S_k > S_{k-1} + R_{th}$, then l is not allocated positive power. However by transmitting on l in $\mathcal{B}$ instead of $\mathcal{N}$, it can be seen that the total throughput will increase, which contradicts this power allocation being optimal. ■

From Prop. 1, the optimal solution to (3) can be found by solving (4) for a given R_{th}, and then iteratively searching for the optimal R_{th}. Solving (4) for a given R_{th} can be done via the following algorithm, which determines the set of channels from $\mathbf{h}$ that are transmitted on.

Algorithm 1 K-best channels $(\mathbf{h}, \mathbf{R_{th}})$

```
initialize:
M = {1, . . . , K}
j = arg max_{i∈M} h_i
W = {j}
d_{-1} = 0
Water-fill over channels {h_i : i ∈ W} giving sum rate d.
if d < R_th then
  W = ∅
else
  while d − d_{-1} > R_th do
    d_{-1} = d
    M = M/W
    j = arg max_{i∈M} h_i
    W = {j} ⋃ W
    Water-fill over channels {h_i : i ∈ W} giving sum rate d.
  end while
end if
return W
```

After each iteration, according to Lemma 1, the rate gain $d - d_{-1}$ decreases. Therefore after at most K steps, the algorithm converges. It converges to the optimal solution of (4). Note a feasible solution might not exist for some channel realizations, in which case the algorithm returns $\mathcal{W} = \emptyset$.

III. Sub-optimal Power Allocation and Asymptotic Analysis

In this section, we introduce a simplified distributed scheme where instead of finding a threshold R_{th} and solving (4), we set a threshold h_{th} on the channel gain. Each user then transmits on the kth channel when its gain is greater than h_{th}, resulting in a transmission probability $p = F_H(h_{th})$. If a user has more than one channel whose gain is higher than the threshold, then the total power P_m will be allocated equally to each of these channels.[2] Given that a user transmitts on k channels it uses a constant rate of $C_i(p) \equiv C\left(F_H^{-1}(p)\frac{P_m}{k}\right)$. This simplified scheme is easy to impliment and analyze. We will show that this simplified scheme is "asymptotically optimal" has defined below.

The total throughput using this scheme is a function of K, N and p. For $i = 1, ..., K$, let $q_{K,p}(i)$ be the probability one user has i channels above the threshold $h_{th} = F_H^{-1}(p)$, where

$$q_{K,p}(i) = \binom{K}{i}(p)^i(1-p)^{K-i}.$$

Among these i channels, for $j = 1, ..., i$, let $\omega_{p,i}(j)$ be the probability a user transmits successfully on j channels, *i.e.* the probability there is no collision on j channels, given i are above the threshold. This is given by

$$\omega_{p,i}(j) = \binom{i}{j}\left[(1-p)^{N-1}\right]^j\left[1-(1-p)^{N-1}\right]^{i-j}.$$

Then the total throughput is given by

$$S(K, N, p) = N\sum_{i=1}^{K} q_{k,p}(i)\sum_{j=1}^{i}\omega_{p,i}(j)jC_i(p).$$

Note $\omega_{p,i}(j)$ is a Binomial distribution and $\sum_{j=1}^{i}\omega_{p,i}(j)j = (1-p)^{N-1}i$. Hence,

$$S(K, N, p) = N(1-p)^{N-1}\sum_{i=1}^{K}\binom{K}{i}(p)^i(1-p)^{K-i}iC_i(p). \tag{5}$$

Recall that $C(x) = \log(1+x)$. Using this the next lemma gives bounds on $S(K, N, p)$,

Lemma 2: For all K, N and p,

$$\log\left(1 + \frac{P_m F_H^{-1}(p)}{p(K-1)+1}\right) \leq \frac{S(K,N,p)}{N(1-p)^{N-1}Kp} \leq \log\left(1 + \frac{P_m F_H^{-1}(p)[1-(1-p)^K]}{Kp}\right).$$

The proof is given in appendix .

Using these bounds, we next consider how the throughput scales in the three regimes given in the introduction. First we introduce the notation $f(x) \precsim g(x)$, which means

[2] This is similar to an approach studied in [8] for a single user channel and in [9] for a downlink OFDM channel.

$\lim_{x\to\infty}\frac{f(x)}{g(x)}=1$, *i.e.* $f(x)$ and $g(x)$ are *asymptotically equivalent*. In each regime, we show that this simplified scheme is asymptotically equivalent to the optimal distributed algorithm. For finite number of users, the throughput gain of the optimal scheme is shown in the simulation result in the last section. We also compare the throughput achieved by the distributed approach to an optimal centralized approach which schedules the users to maximize the throughput in any slot (still assuming at most one user can transmit in each channel). This is given by:[3]

$$\begin{aligned}&\max_{\{P_{n,k},c_{n,k}\}}\sum_n\sum_k C(P_{nk}c_{nk}h_{nk})\\ &\text{s.t. }\sum_k P_{nk}c_{nk}=P_m,\forall\, c_{nk}=\{0,1\},\ \sum_n c_{nk}=1.\end{aligned} \tag{6}$$

Let $S_{ct}(K,N)$ be the average throughput obtained by this scheduling policy. Denote the throughput of the optimal distributed policy by $S^*(K,N)$ (i.e. the policy in Prop. 1) and the optimal throughput of the threshold-based algorithm as $S(K,N,p^*)$, where p^* is the transmission probability that optimizes $S(K,N,p)$. Note that for all N and K,

$$S\left(K,N,\tfrac{1}{N}\right)\le S(K,N,p^*)\le S^*(K,N)\le S_{ct}(K,N), \tag{7}$$

where the first term is the throughput with a transmission probability of $1/N$. In the following discussions, we will give results for a Rayleigh fading channel, where $F_H(h)=\exp(-h/h_0)$. Extensions can be made to fading distribution that satisfy certain restrictions as in [6].

First we consider the case where K is fixed and N increases.

Proposition 2: Given any finite K, as $N\to\infty$, $S(K,N,\frac{1}{N})$, $S(K,N,p^*)$, $S^*(K,N)$ and $\frac{1}{e}S_{ct}(K,N)$ are all asymptotically equivalent to $\frac{K}{e}\log\left(1+P_m h_0\log(N)\right)$.

In other words, asymptotically there is no loss in performance from using the threshold based scheme or from choosing $p=\frac{1}{N}$. The throughput in each case is asymptotically increases like $\frac{K}{e}\log\left(1+P_m h_0\log(N)\right)$. Also, the ratio of the throughput of each approach compared to that of a centralized scheduler is asymptotically equal to $\frac{1}{e}$.

Proof: From (7), to prove that $S\left(K,N,\frac{1}{N}\right)$, $S(K,N,p^*)$, and $S^*(K,N)$ are all asymptotically equivalent to $\frac{K}{e}\log\left(1+P_m h_0\log(N)\right)$, it is sufficient to show that

$$\lim_{N\to\infty}\frac{S(K,N,\frac{1}{N})}{\frac{K}{e}\log\left(1+P_m h_0\log(N)\right)}\ge 1,$$

and

$$\lim_{N\to\infty}\frac{S^*(K,N)}{\frac{K}{e}\log\left(1+P_m h_0\log(N)\right)}\le 1.$$

[3]This is similar to a problem studied for centralized OFDM systems in [5].

From Lemma 2,

$$\begin{aligned}&S(K,N,\tfrac{1}{N})\\ &\ge(1-N^{-1})^{N-1}K\log\left(1+\frac{P_m h_0\log(N)}{N^{-1}(K-1)+1}\right).\end{aligned}$$

Letting $S_l(K,N,\frac{1}{N})$ denote this lower bound, it can be seen that

$$\lim_{N\to\infty}\frac{S_l(K,N,\frac{1}{N})}{\frac{K}{e}\log\left(1+P_m h_0\log(N)\right)}=1. \tag{8}$$

Therefore, we have

$$\lim_{N\to\infty}\frac{S(K,N,\frac{1}{N})}{\frac{K}{e}\log\left(1+P_m h_0\log(N)\right)}\ge 1.$$

Now consider a model with K parallel channels, where each user has peak power P_m for each channel, instead of having the total power across all the channels constrained by P_m. Other assumptions remain the same. Denote the optimal throughput of the parallel model as $S_p(K,N)$, and $S_p(K,N)=KS_p(1,N)$. In [6], we proved that for a single channel $S_p(1,N)\asymp\frac{1}{e}\log(1+P_m h_0\log(N))$. So we have $S^*(K,N)\le\frac{K}{e}\log(1+P_m h_0\log(N))$.

Next, we show that $S(K,N,\frac{1}{N})\asymp\frac{1}{e}S_{ct}(K,N)$. Consider a centralized parallel channel model, where the user who has the best channel to transmit at each channel and users are not constrained by total power P_m, *i.e.* each transmission will use power P_m, if one user transmit at more than one channel, the power constraint is violated. The throughput of this model the upper bound of $S_{ct}(K,N)$, denoted as $S_{ctu}(K,N)$. Again in [6] for a single channel, it is shown that $\lim_{N\to\infty}\frac{S_{ctu}(K,N)}{K\log(1+P_m h_0\log(N))}=1$. Therefore $\lim_{N\to\infty}\frac{S(K,N,\frac{1}{N})}{S_{ct}(K,N)}\ge\frac{1}{e}$. Now consider a lower bound of $S_{ct}(K,N)$. The same as the parallel model, we chooose the user who has the best channel to transmit at each channel. However, if one user is chosen for more than one channel, it will split the power and allocate equal power to each channel. Denote the throughput of this model as $S_{ctl}(K,N)$, which is a lower bound of $S_{ct}(K,N)$. This is given by

$$\begin{aligned}&S_{ctl}(K,N)=\\ &N\sum_{i=1}^{K}\binom{K}{i}(\frac{1}{N})^i(1-\frac{1}{N})^{K-i}i\mathbb{E}\left(\log(1+\frac{P_m H_{max}}{i})\right).\end{aligned}$$

In [6] it is shown that

$$\lim_{N\to\infty}\frac{\mathbb{E}\left(\log(1+\frac{P_m H_{max}}{i})\right)}{\log\left(1+\frac{P_m h_0\log(N)}{i}\right)}=1.$$

Combining this with (5), we have

$$\lim_{N\to\infty}\frac{S(K,N,\frac{1}{N})}{S_{ctl}(K,N)}=\frac{1}{e}.$$

Hence,

$$\lim_{N\to\infty} \frac{S(K,N,\frac{1}{N})}{S_{ct}(K,N)} = \frac{1}{e}.$$

■

Next, we consider the case, where N is fixed, and K increases.

Proposition 3: Given any finite N, as $K \to \infty$, $S(K,N,p^*)$, $S^*(K,N)$, $S_{ct}(K,N)$ are all asymptotically equivalent to $NP_m h_0 \log(K)$.

Proof: Note for all K and N, $S\left(K,N,\frac{1}{N}\right) \le S(K,N,p^*) \le S^*(K,N) \le S_{ct}(K,N)$, it is enough to show that

$$\lim_{N\to\infty} \frac{S_{ct}(K,N)}{NP_m h_0 \log(K)} \le 1,$$

and

$$\lim_{N\to\infty} \frac{S(K,N,\frac{1}{N})}{NP_m h_0 \log(K)} \ge 1.$$

Shown in [8], the optimal throughput for one user, $S_{ct}(K,1)$ satisfies,

$$\lim_{K\to\infty} \frac{S_{ct}(K,1)}{P_m h_0 \log K} = 1.$$

With N users, the maximum throughput could be no more than $NS_{ct}(K,1)$, therefore

$$\lim_{N\to\infty} \frac{S_{ct}(K,N)}{NP_m h_0 \log(K)} \le 1.$$

Choose $p = \frac{f(K)}{K}$, where $f(K)$ satisfies $\lim_{K\to\infty} \frac{f(K)}{K} = 0$ and $\lim_{K\to\infty} \frac{\log(K)}{f(K)} = 0$. Therefore the increasing rate of $f(K)$ with K is slower than K, but faster than $\log(K)$. One example of $f(K)$ could be $f(K) = (\log(K))^2$.

Then we have

$$S_l(K,N,p) = N(1-p)^{N-1} Kp \log\left(1 - \frac{P_m h_0 \log(p)}{p(K-1)+1}\right),$$

thus

$$S_l(K,N,\frac{f(K)}{K}) = N\left(1 - \frac{f(K)}{K}\right)^{N-1} f(K) \log\left(1 - \frac{P_m h_0 \log(\frac{f(K)}{K})}{f(K)\frac{K-1}{K}+1}\right).$$

Because $\lim_{K\to\infty} \frac{f(K)}{K} = 0$, we have $\lim_{K\to\infty}\left(1 - \frac{f(K)}{K}\right)^{N-1} = 1$.

Also because $\lim_{K\to\infty} \frac{\log(K)}{f(K)} = 0$, we have

$$\lim_{K\to\infty} \frac{f(K)\log\left(1 - \frac{P_m h_0 \log(\frac{f(K)}{K})}{f(K)}\right)}{P_m h_0 \log(K)} = 1.$$

Therefore, we have

$$\lim_{K\to\infty} \frac{S_l(K,N,\frac{f(K)}{K})}{NP_m h_0 \log(K)} = 1,$$

and

$$\lim_{N\to\infty} \frac{S(K,N,\frac{1}{N})}{NP_m h_0 \log(K)} \ge 1.$$

■

This implies that again the threshold based approach is asymptotically equivalent to the optimal distributed approach. In this case there is no asymptotic loss compared to the centralized approach. This is because as the number of channels increases the probability of collision becomes negligible. For a Rayleigh fading channel each of these terms grows like $\log(K)$ as $K \to \infty$. Finally, we let both K and N increase with fixed ratio $\frac{K}{N} = \beta$,

Proposition 4: If $\frac{K}{N} = \beta$, as $N \to \infty$, $S(\beta N, N, \frac{1}{N})$, $S(\beta N, N, p^*)$, $S^*(\beta N, N)$ and $\frac{1}{e} S_{ct}(\beta N, N)$ are all asymptotically equivalent to $\beta N e^{-1} \log\left(1 + P_m h_0 \log(N)\right)$.

Proof uses similar ideas to Prop. 2. See appendix .

As in Prop. 1, once again compared to the centralized scheme there is an asymptotic penalty of $1/e$ due to the contention, and the throughput grows like $\log(\log(N))$.

Next we compare the distributed approach to several schemes that require minimal coordination for assigning different users different power allocation policies. First assume that $\frac{K}{N} = \beta$, where β is a positive integer. In this case a "non-collision scheme" is to assign β channels to each user for all time. The throughput of this scheme $S_{nc}(K,N) = NS_{ct}(\beta,1)$. Compared to $S(K,N)$, we have

$$\lim_{N\to\infty} \frac{S_{nc}(\beta N,N)}{S(\beta N,N)} = \lim_{N\to\infty} \frac{S_{ct}(\beta,1)}{\beta e^{-1}\log(1+P_m h_0 \log(N))} = 0.$$

This is because the non-collision scheme can not exploit any "multi-user" diversity. However, if N is fixed and let β increases, *i.e.* K increases, because $S_{ct}(\beta,1) \asymp P_m h_0 \log(\beta)$, we have $S_{nc}(K,N) \asymp NP_m h_0 \log(\beta) \asymp NP_m h_0 \log(K)$. In this case, both approaches see increased frequency diversity as K increases.

Next assume $\alpha = \frac{N}{K} > 1$; in this case an approach with fewer collisions is to assign α users to each channel for all time. This results in a throughput of $S_{lc}(K,N) = KS^*(1,\alpha)$. It follows that

$$\lim_{N\to\infty} \frac{S_{lc}(K,N)}{S(K,N,p^*)} = \lim_{N\to\infty} \frac{S^*(1,\alpha)}{e^{-1}\log(1+P_m h_0 \log(N))} = 0.$$

Again, the new scheme cannot fully exploit the available diversity. However, if α increases for fixed K, because $S_{lc}(1,N) \asymp \frac{1}{e} P_m h_0 \log(\log\alpha)$, we have $S_{lc}(K,N) \asymp \frac{K}{e} P_m h_0 \log(\log\alpha) \asymp \frac{K}{e} P_m h_0 \log(\log(N))$. The throughput ratio of the two schemes approaches to 1 asymptotically, because both exploits multiuser diversity.

IV. Numerical Examples

To conclude we give some numerical examples. Fig. 3 shows the throughput gain achieved by the optimal power allocation scheme compared to the simplified power allocation scheme. As the number of users increases, the throughput gain decreases. Fig. 4 shows the upper and lower bounds of the throughput ratio of the optimal distributed scheme $S^*(K,N)$ to the centralized scheme $S_{ct}(K,N)$ defined in (6). Calculating $S_{ct}(K,N)$ requires solving the optimization problem in (6), which is complicated due to the integer constraints. Instead we compare $S^*(K,N)$ to the upper and lower bounds of $S_{ct}(K,N)$. Fig. 4 shows the two bounds for the throughput ratio for fixed number of channels as the number of users increases. Here we obtain the upper bound of $S_{ct}(K,N)$ by relaxing the total power constraint on the channels $\sum_k P_{nk}c_{nk} = P_m$. Instead, we allow each user transmit with $P_{nk} = P_m$ over each channel. The maximal throughput is then achieved for this relaxing system by letting the best user of each channel to transmit at each time. We take this maximal throughput as our upper bound. To lower bound $S_{ct}(K,N)$, we still choose the best user to transmit on each channel, but if one user is chosen to transmit on more than one channels. The power P_m is divided equally to these channels. The achievable throughput is then a lower bound of $S_{ct}(K,N)$. Fig. 4 shows that as the number of users increases, the two bounds comes closer. The reason is that the probability that one user is chosen to transmit at more than one channel is small for a larger number of users. It can be seen that the actual throughput ratio is decreasing as the number of users increases, and is higher than the contention factor $1/e$ inherent in an Aloha system. Therefore the actual average transmission rate is higher than the average transmission rate for a centralized system. The same behavior has been observed in the single channel case in [6]. Also as the number of the channels K increases, a throughput gain is achieved by exploiting frequency diversity. Fig. 5 shows the upper and lower bounds for the throughput ratio for fixed number of users as the number of channels increases. In this case, we upper bound $S_{ct}(K,N)$ by the information theoretic capacity of this multi-access system. In other words, joint decoding is used when a collision happens. We use the iterative water-filling algorithm [4] to obtain the upper bound. One channel can be assigned to multiple users to achieve the capacity. By only allowing the user who has the best channel to transmit on that channel, we obtain a lower bound of the system. Fig. 5 shows as the number of channels increases, two bounds becomes closer. The throughput ratio increases as the number of channels increases.

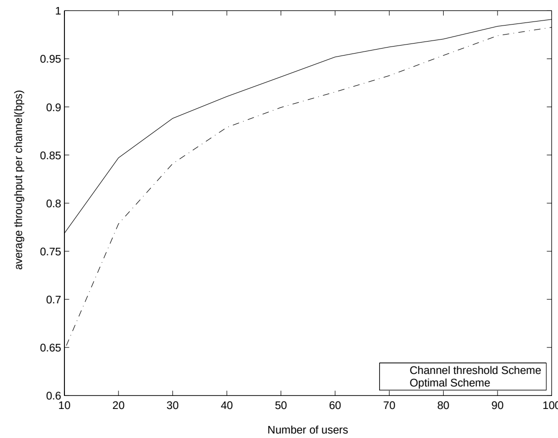

Fig. 3. Average throughput (bps) per channel for $K = 10$ versus number of users. Optimal scheme with optimal rate threshold and simplified scheme with channel gain threshold, both schemes use water-filling power allocation over the channels selected.

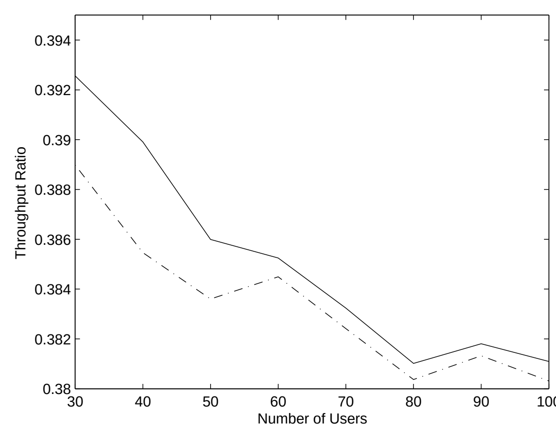

Fig. 4. Lower and Upper bounds of the throughput ratio of the optimal distributed scheme to centralized scheme versus number of users for $K = 10$.

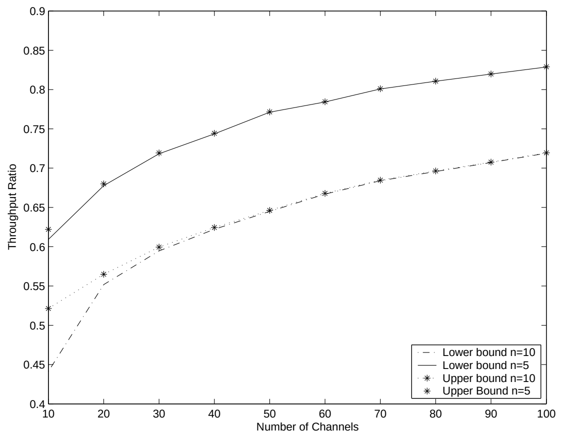

Fig. 5. Lower and Upper bounds of the throughput ratio of the optimal distributed scheme to centralized scheme versus number of channels for $N = 5$ and $N = 10$.

APPENDIX

Proof: Let

$$\begin{aligned}
R_K =& \sum_{i=1}^{K} \binom{K}{i} (p)^i (1-p)^{K-i} i \log\left(1 + \frac{P_m F_H^{-1}(p)}{i}\right) \\
=& \sum_{i=1}^{K} \frac{K!}{(K-i)!(i-1)!} p^i (1-p)^{K-i} \\
& \log\left(1 + \frac{P_m F_H^{-1}(p)}{i}\right) \\
=& K \sum_{j=0}^{K-1} \frac{(K-1)!}{(K-1-j)!j!} p^{j+1} (1-p)^{K-j-1} \\
& \log\left(1 + \frac{P_m F_H^{-1}(p)}{j+1}\right) \\
=& Kp \sum_{j=0}^{K-1} \binom{K-1}{j} p^j (1-p)^{K-1-j} \\
& \log\left(1 + \frac{P_m F_H^{-1}(p)}{j+1}\right)
\end{aligned}$$

The function $g(x) = \log(1 + \frac{P_m F_H^{-1}(p)}{x+1})$ can be shown to be a convex function, for $x > 0$. Also

$$\Pi_j = \binom{K-1}{j} p^j (1-p)^{K-1-j}, j = 0..K-1$$

is a pmf. Hence, using Jenson's inequality, we have

$$R_K \geq Kp \log\left(1 + \frac{P_m F_H^{-1}(p)}{p(K-1)+1}\right).$$

Therefore,

$$S_k \geq N(1-p)^{N-1} Kp \log\left(1 + \frac{P_m F_H^{-1}(p)}{p(K-1)+1}\right),$$

which gives the desired lower bound.

Next, we derive the upper bound. The function $f(i) = i\log(1+\frac{P_m F_H^{-1}(p)}{i})$ can be shown to be a concave function, for $i > 0$. Hence, we have

$$\begin{aligned}
R_K &= \sum_{i=1}^{K} \binom{K}{i} (p)^i (1-p)^{K-i} f(i) \\
&= \frac{\sum_{i=1}^{K} \binom{K}{i} (p)^i (1-p)^{K-i} f(i)}{\sum_{i=1}^{K} \binom{K}{i} (p)^i (1-p)^{K-i}} \\
& \sum_{i=1}^{K} \binom{K}{i} (p)^i (1-p)^{K-i}.
\end{aligned}$$

Taking expectation with respect to pmf

$$\Pi_i = \frac{\binom{K}{i} (p)^i (1-p)^{K-i}}{\sum_{i=1}^{K} \binom{K}{i} (p)^i (1-p)^{K-i}}, i = 1..K,$$

and again using Jenson's inequality, we have

$$\begin{aligned}
R_K &= \mathbb{E} f(i) [1 - (1-p)^K] \\
&\leq f(\mathbb{E} i) [1 - (1-p)^K] \\
&= f\left(\frac{Kp}{1-(1-p)^K}\right) [1 - (1-p)^K] \\
&= Kp \log\left(1 + \frac{P_m F_H^{-1}(p)[1-(1-p)^K]}{Kp}\right).
\end{aligned}$$

Therefore, we have the upper bound

$$S_K \leq N(1-p)^{N-1} Kp \log(1 + \frac{P_m F_H^{-1}(p)[1-(1-p)^K]}{Kp}).$$

■

Proof: Note for all N, to prove that $S\left(\beta N, N, \frac{1}{N}\right)$, $S(\beta N, N, p^*)$, and $S^*(\beta N, N)$ are all asymptotically equivalent to $\frac{\beta N}{e} \log\left(1 + P_m h_0 \log(N)\right)$, it is sufficient to show that

$$\lim_{N\to\infty} \frac{S(\beta N, N, \frac{1}{N})}{\frac{\beta N}{e} \log\left(1 + P_m h_0 \log(N)\right)} \geq 1,$$

and

$$\lim_{N\to\infty} \frac{S^*(\beta N, N)}{\frac{\beta N}{e} \log\left(1 + P_m h_0 \log(N)\right)} \leq 1.$$

From Lemma 2,

$$\begin{aligned}
& S_l(\beta N, N, \frac{1}{N}) \\
& = (1 - N^{-1})^{N-1} \beta N \log(1 + \frac{P_m h_0 \log(N)}{N^{-1}(\beta N - 1) + 1}),
\end{aligned}$$

$$\lim_{N\to\infty} \frac{S_l(\beta N, N, \frac{1}{N})}{\frac{\beta N}{e} \log\left(1 + P_m h_0 \log(N)\right)} = 1. \quad (9)$$

Therefore, we have

$$\lim_{N\to\infty} \frac{S(\beta N, N, \frac{1}{N})}{\frac{\beta N}{e}\log(1+P_m h_0 \log(N))} \geq 1.$$

Now consider a model with K parallel channels, where each user has peak power P_m for each channel same as proof of Prop. 2. Denote the optimal throughput of the parallel model as $S_p(K,N)$, $S_p(K,N) \geq S^*(K,N)$ and $S_p(K,N) = KS_p(1,N)$. In [6], we proved that $S_p(1,N) \precsim \frac{1}{e}\log(1 + P_m h_0 \log(N))$. So we have $S^*(\beta N, N) \leq \frac{\beta N}{e}\log(1+P_m h_0 \log(N))$.

With the same argument as proof of Prop. 2, we also have $S(K,N,\frac{1}{N}) \precsim \frac{1}{e}S_{ct}(K,N)$. ■

In this paper, we proposed a distributed channel-aware Aloha for multiple channel model. The optimal distributed algorithm is given and the simplified algorithm is analyzed and asymptotic results are shown for three different cases. The centralized optimal problem is formulated. And simulation results are shown with comparison to the optimal solution.

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