Reduction of NP Problems
&
Property-Based Testing

Chenhao Zhang

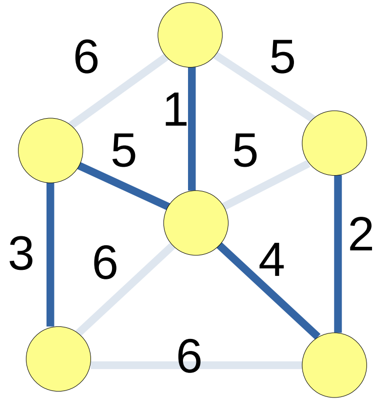
CS396 Spring 2023
Northwestern

Plan of the week

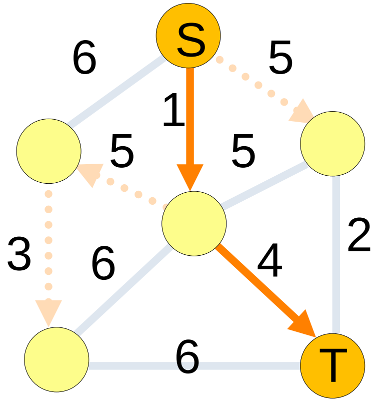
- **NP Problem & Reduction (Today)**
- Examples, Reduction in Karp -- Wednesday
- Lab, Assignment 4 -- Friday

Many problems have efficient algorithms

Minimum Spanning Tree



Shortest path



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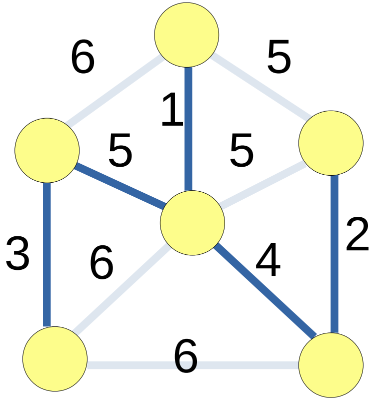

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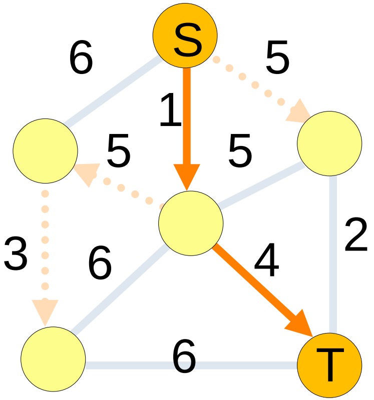

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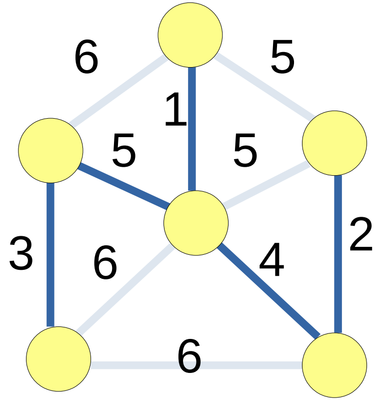
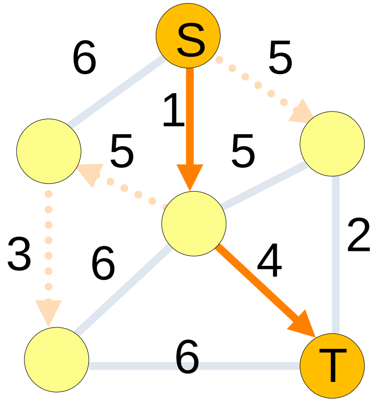
version with Yes/No answer

Has Spanning Tree w/ Cost <=15 ?



Has S-T path w/ Cost <=5 ?



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version with Yes/No answer – *decision problem*

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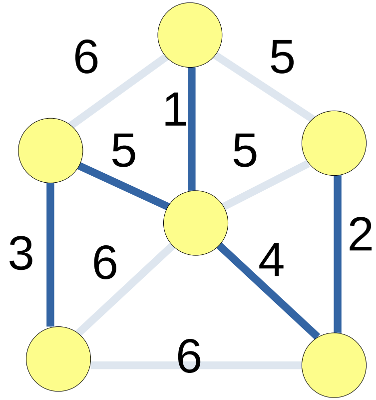

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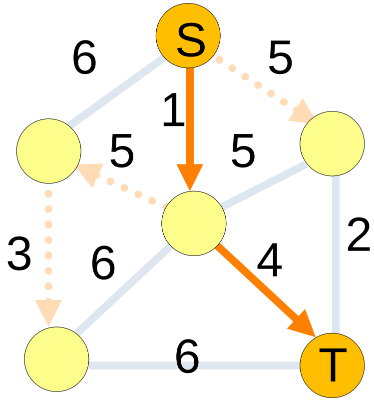

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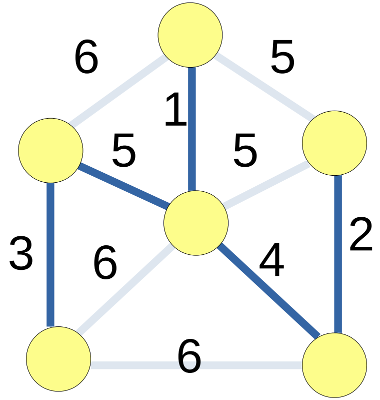

1+5+3+4+2=15

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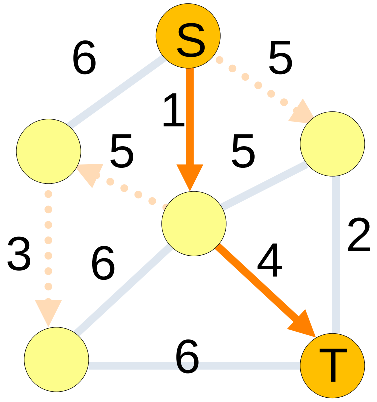

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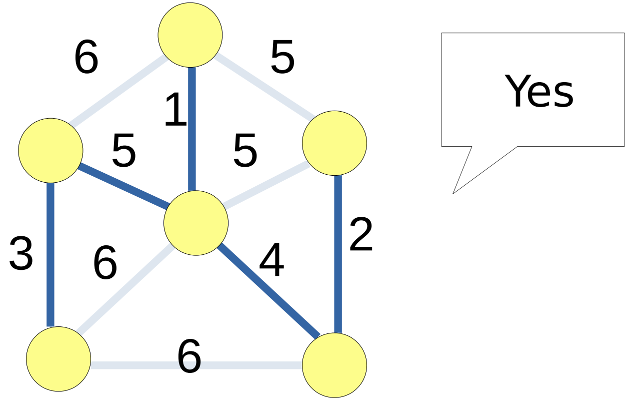

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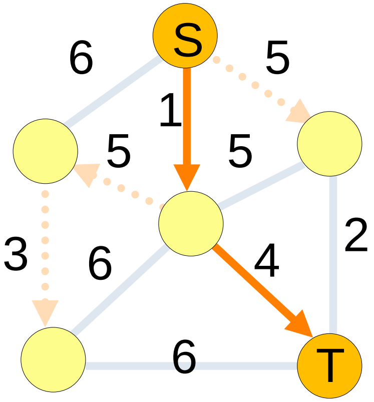

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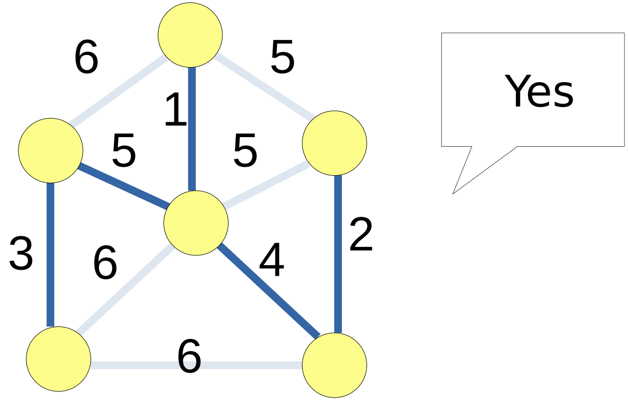

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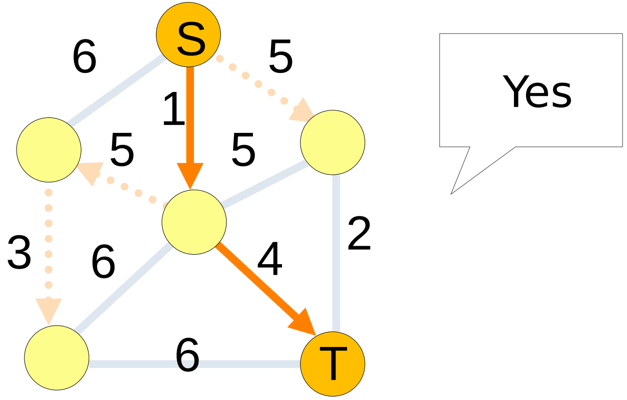

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Yes-Instance has a *certificate*, i.e., proof of yes

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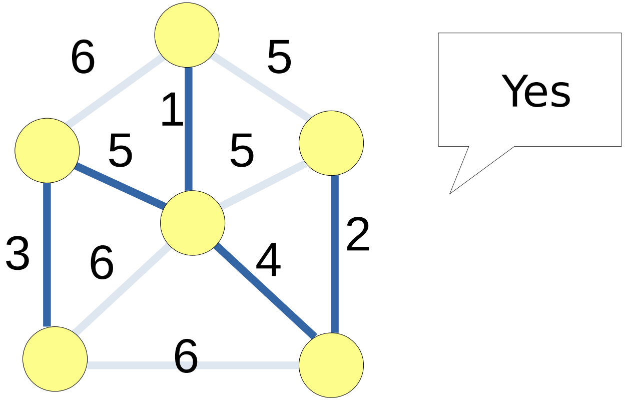

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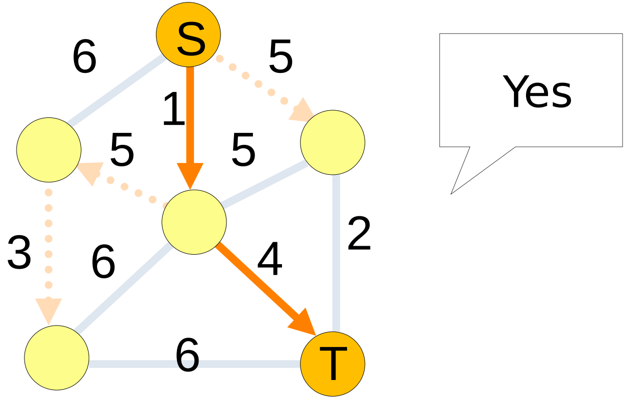

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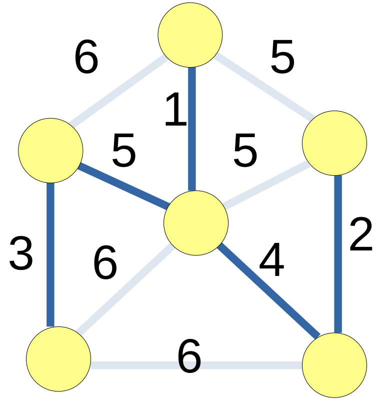
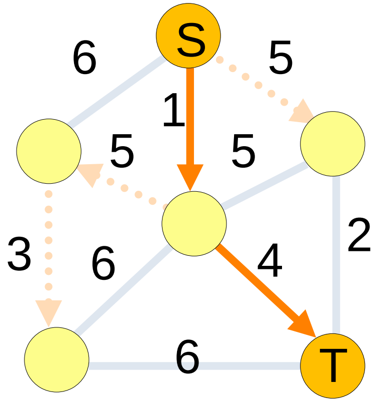
No-Instance has no *certificate*, ~~proof of yes~~

Has Spanning Tree w/ Cost <=**14** ?



1+5+3+4+2=15 > 14

Has S-T path w/ Cost <=**4** ?



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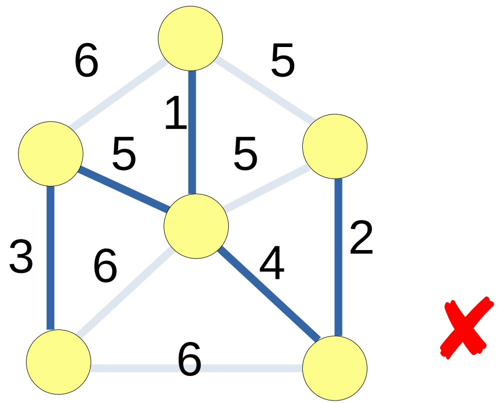

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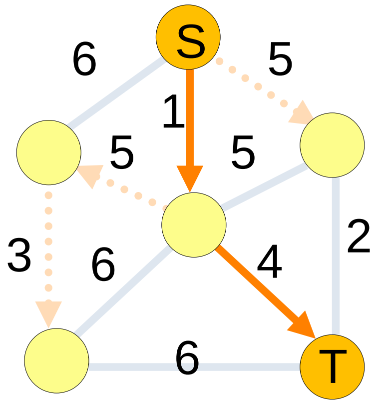

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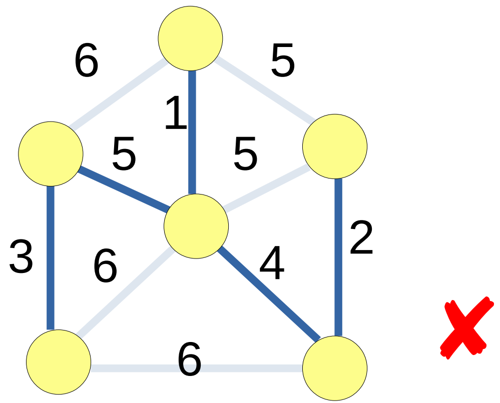

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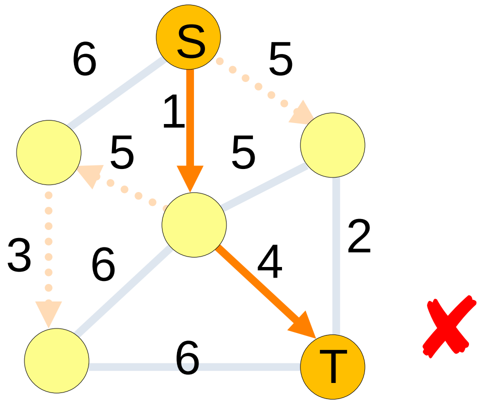

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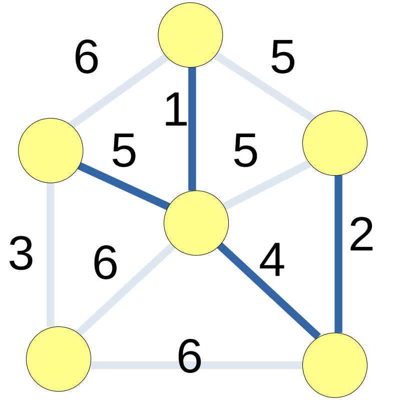

1+5+4+2=12 <= 14

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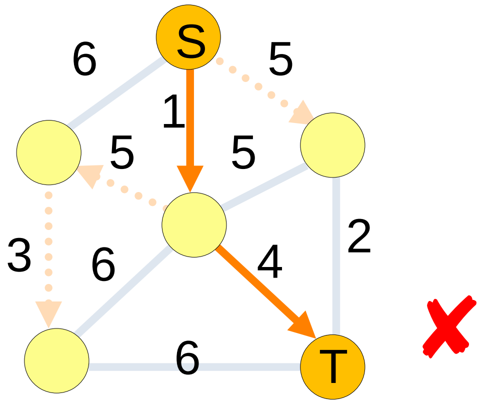

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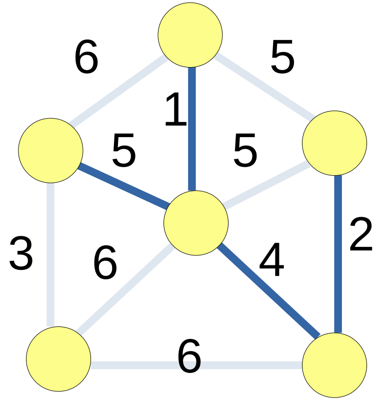

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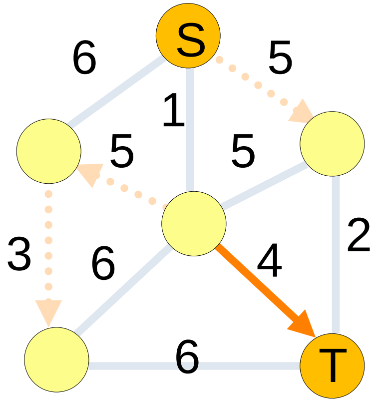

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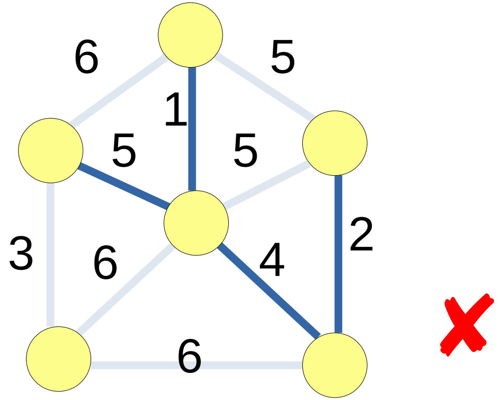

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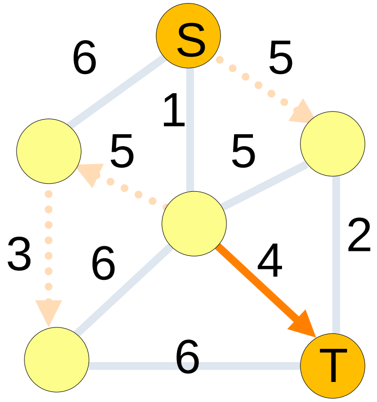

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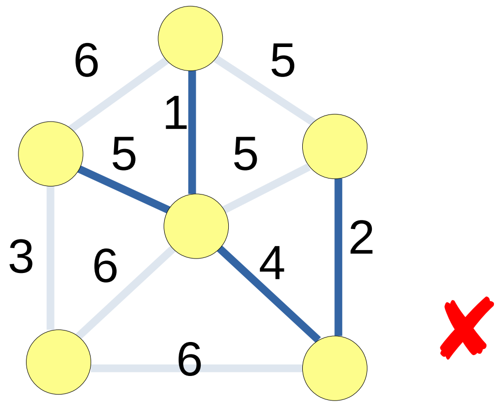

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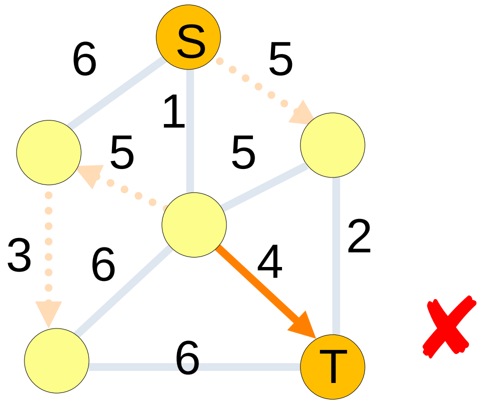

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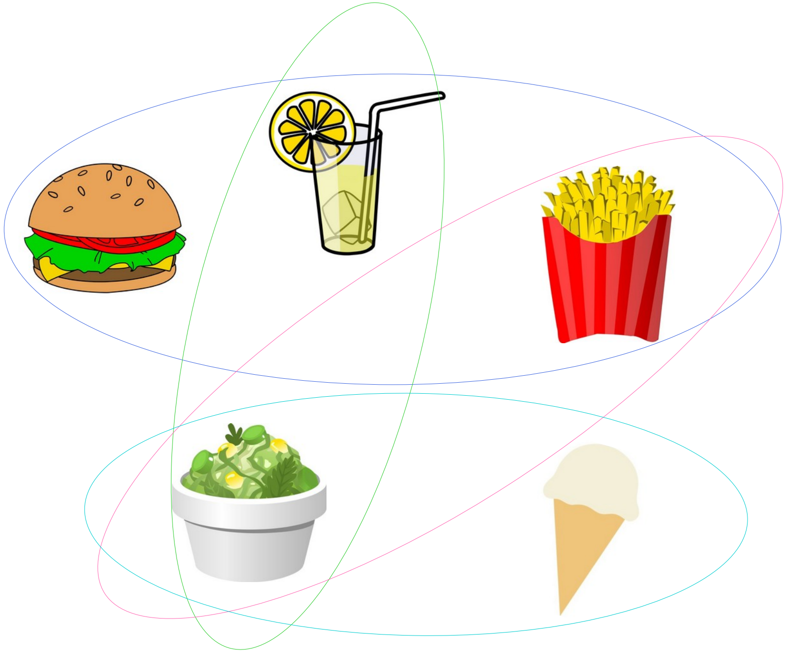
Can we get all by buying only **2** bundles?



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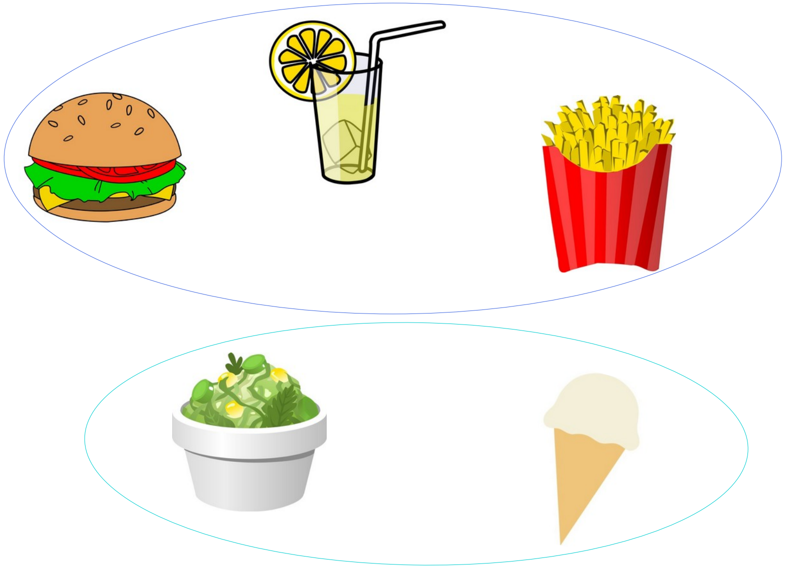

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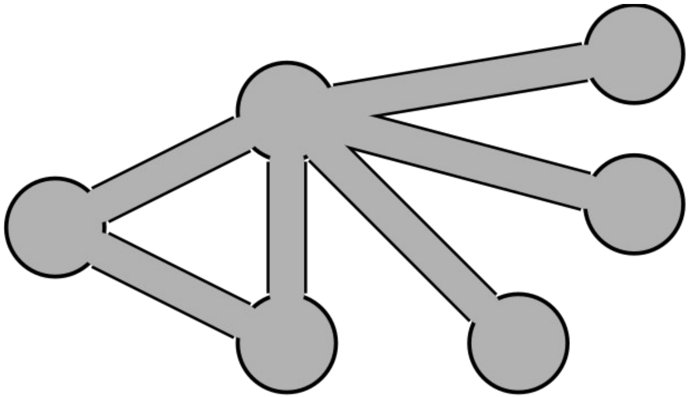

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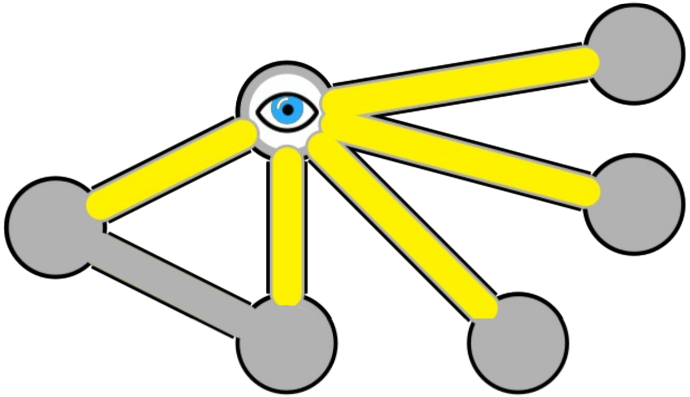

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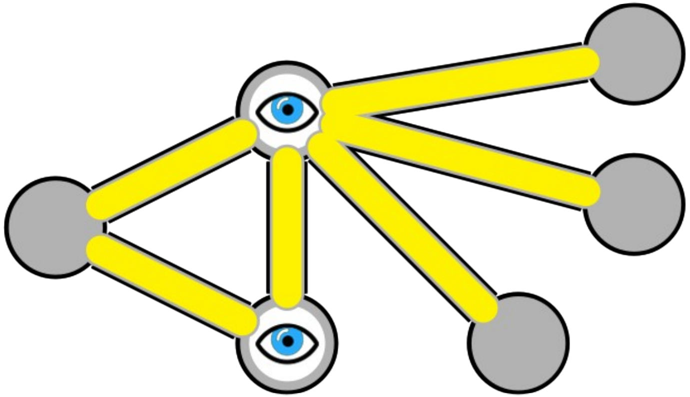

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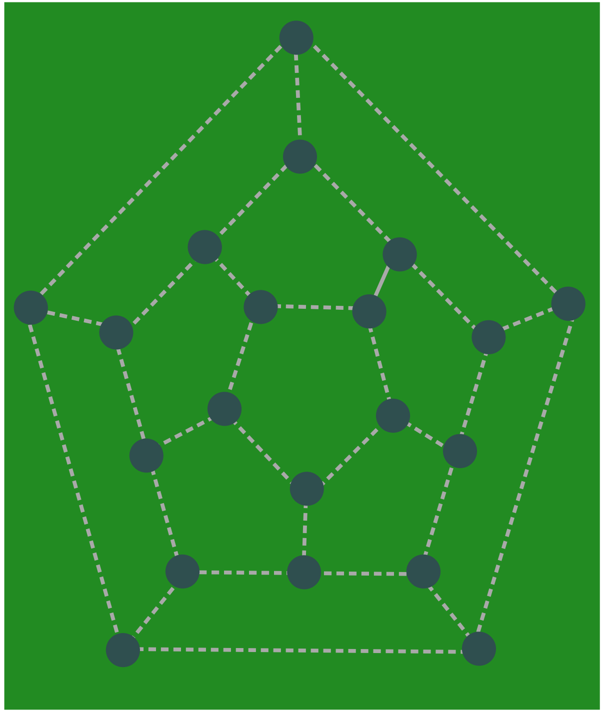

HAMILTONIAN-CYCLE

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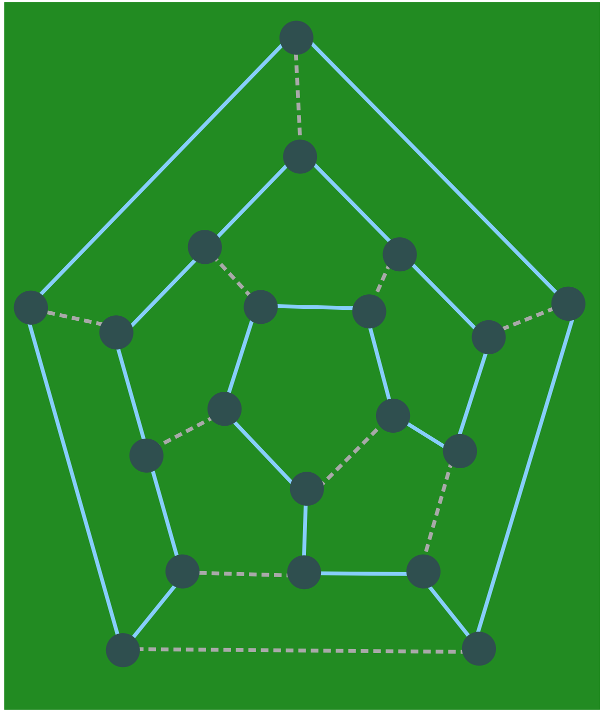
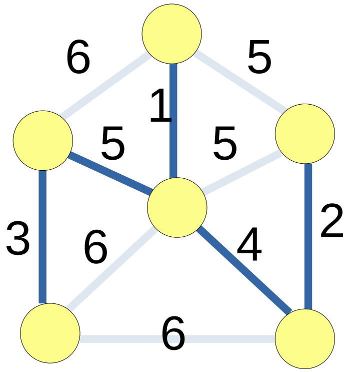
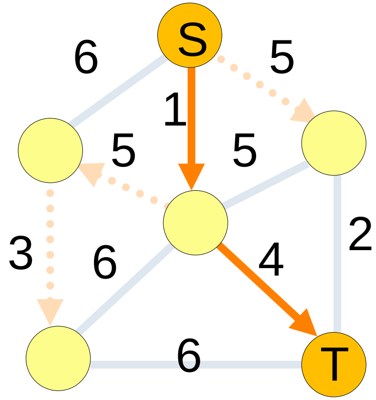


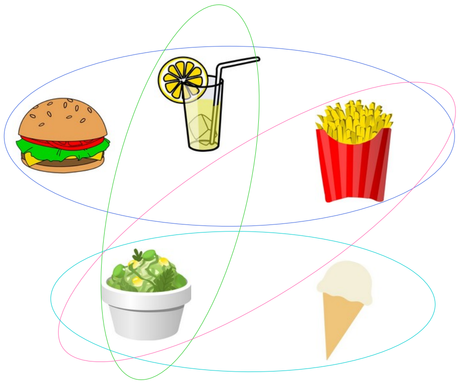
HAMILTONIAN-CYCLE



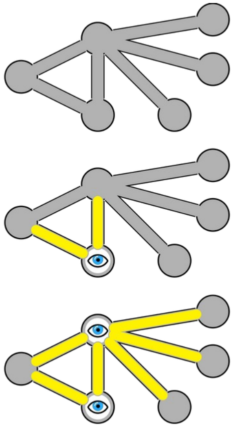
Minimum-Spanning-Tree



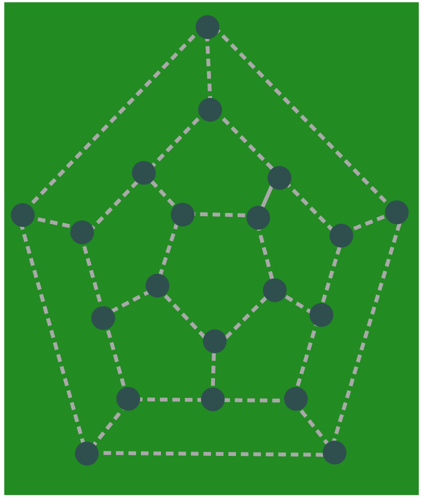
Shortest-Path



Set-Cover



Vertex-Cover



Hamiltonian-Cycle

Q: What do they have in common?

SET-COVER

VERTEX-COVER

HAMILTONIAN-CYCLE

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(can be done in polynomial-time)

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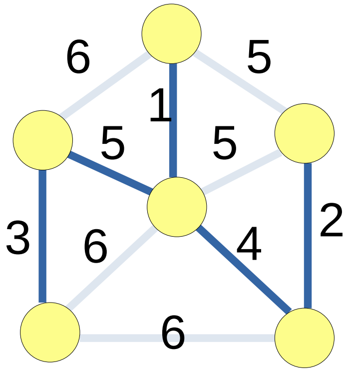
NP-Problems

(**N**on-deterministic **P**olynomial-time)

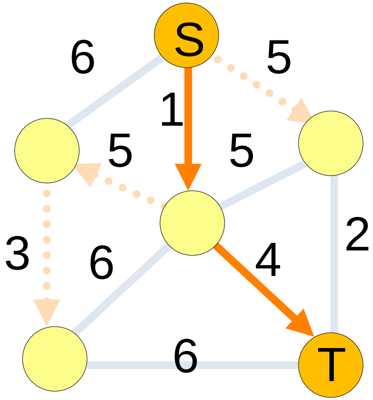
SET-COVER

VERTEX-COVER

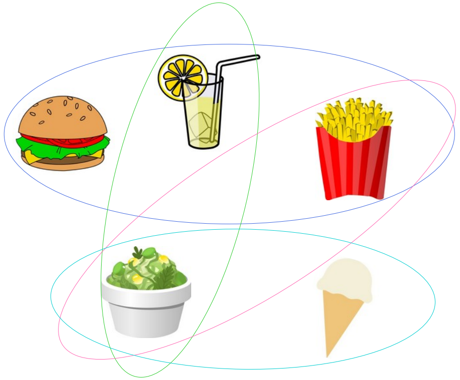
HAMILTONIAN-CYCLE



MINIMUM-SPANNING-TREE



SHORTEST-PATH



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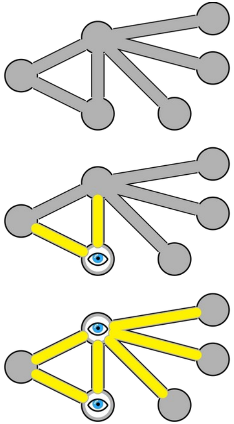
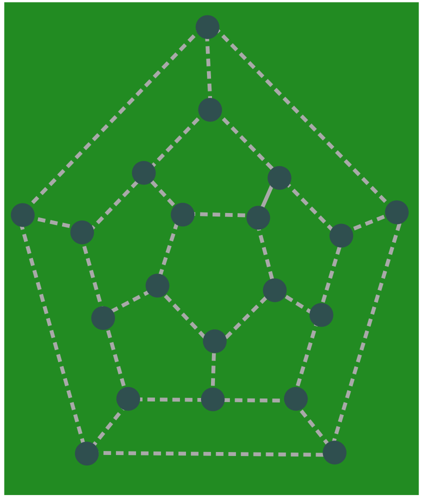


VERTEX-COVER



HAMILTONIAN-CYCLE

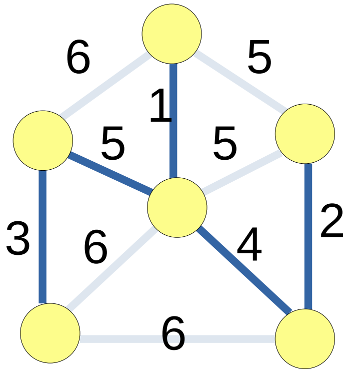
Q: Any difference?

SET-COVER

VERTEX-COVER

HAMILTONIAN-CYCLE

“Easy”



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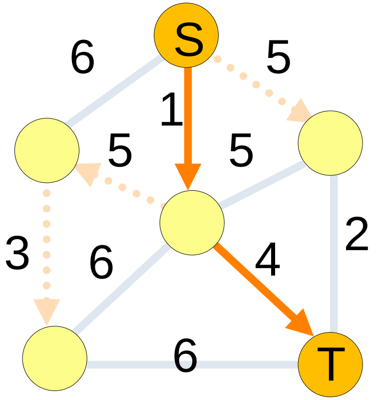
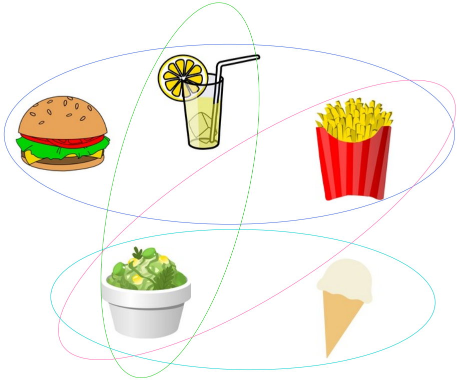


SHORTEST-PATH

“Hard”



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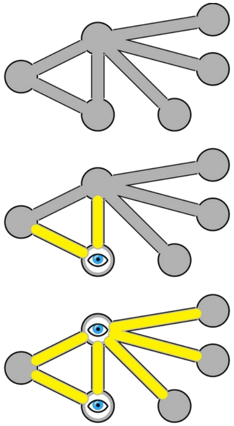

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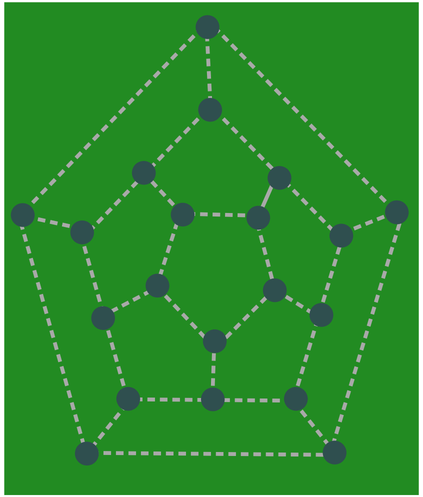

HAMILTONIAN-CYCLE

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“Hard” problems have NO efficient algorithms

HAMILTONIAN-CYCLE

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But there’s no proof for it yet...

Set-Cover Vertex-Cover Hamiltonian-Cycle

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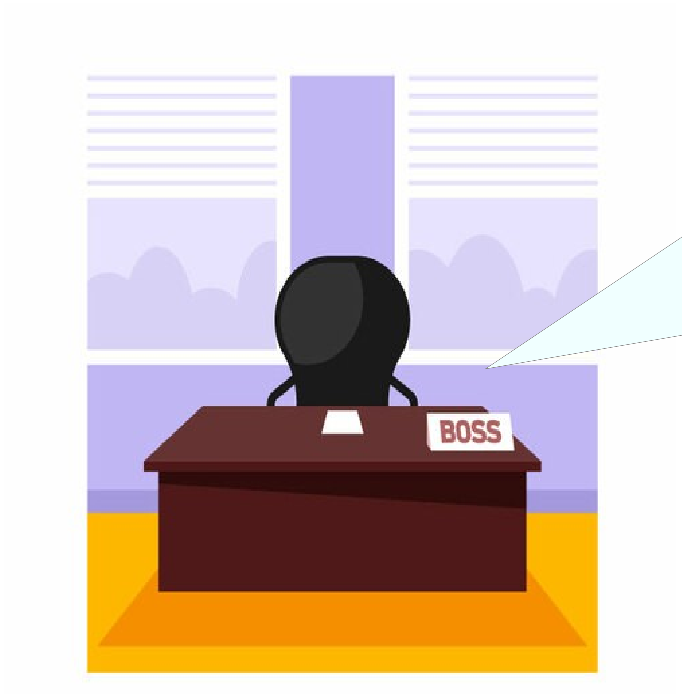
SET-COVER VERTEX-COVER HAMILTONIAN-CYCLE

How do you prove that an NP-problem is "Hard"?



Design an efficient algorithm for problem N!

How do you prove that an NP-problem is “Hard”?

Design an efficient algorithm for problem N!

But… problem N is “Hard”

How do you prove that an NP-problem is “Hard”?

If N could be solved,
a known hard problem H
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How do you prove that an NP-problem is "Hard"?

"reduction"

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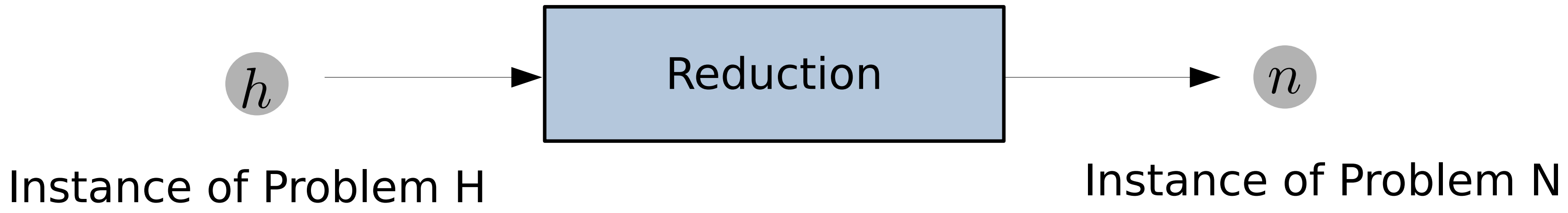
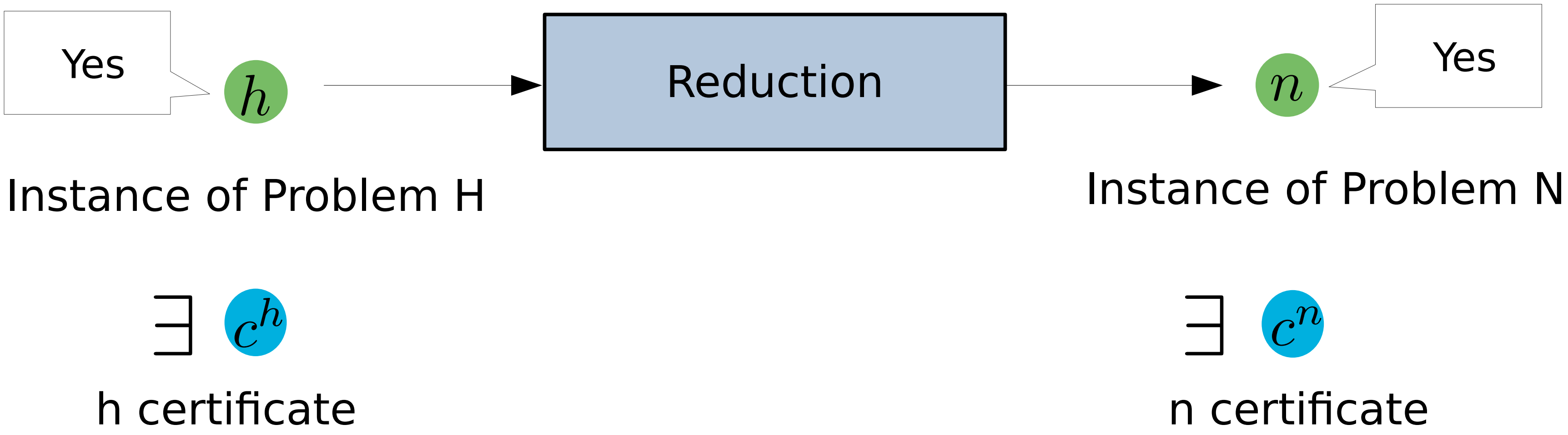
One-Call Reduction



One-Call Reduction – Correctness Property

H is the problem known to be hard

n is the new problem



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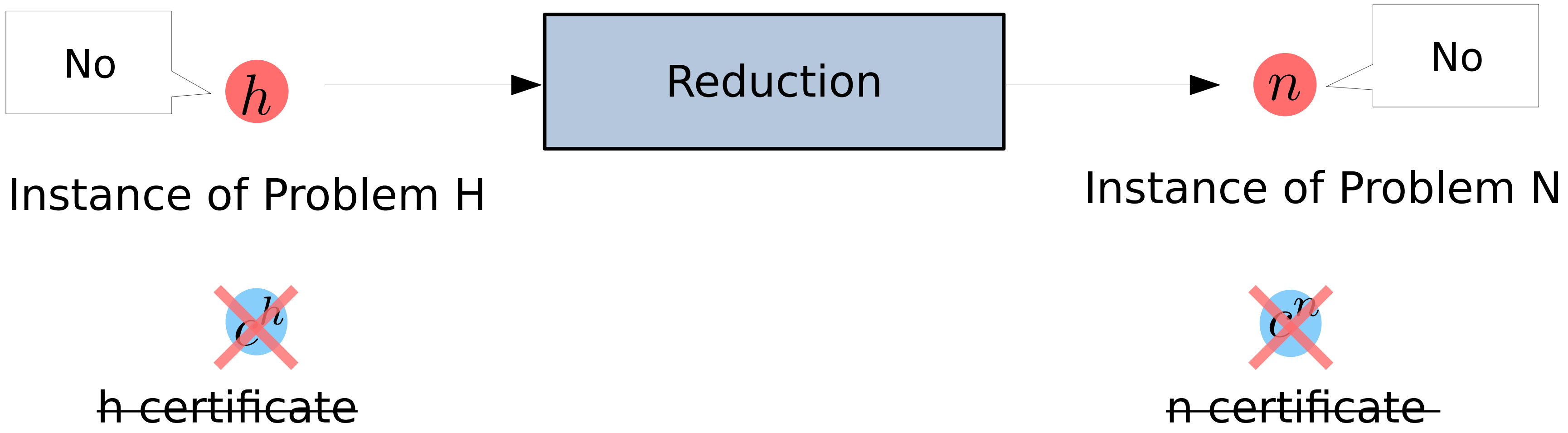

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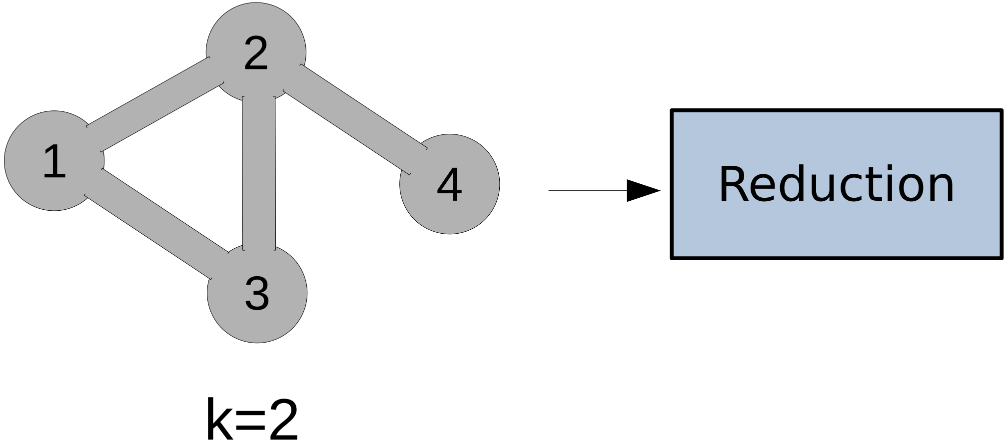
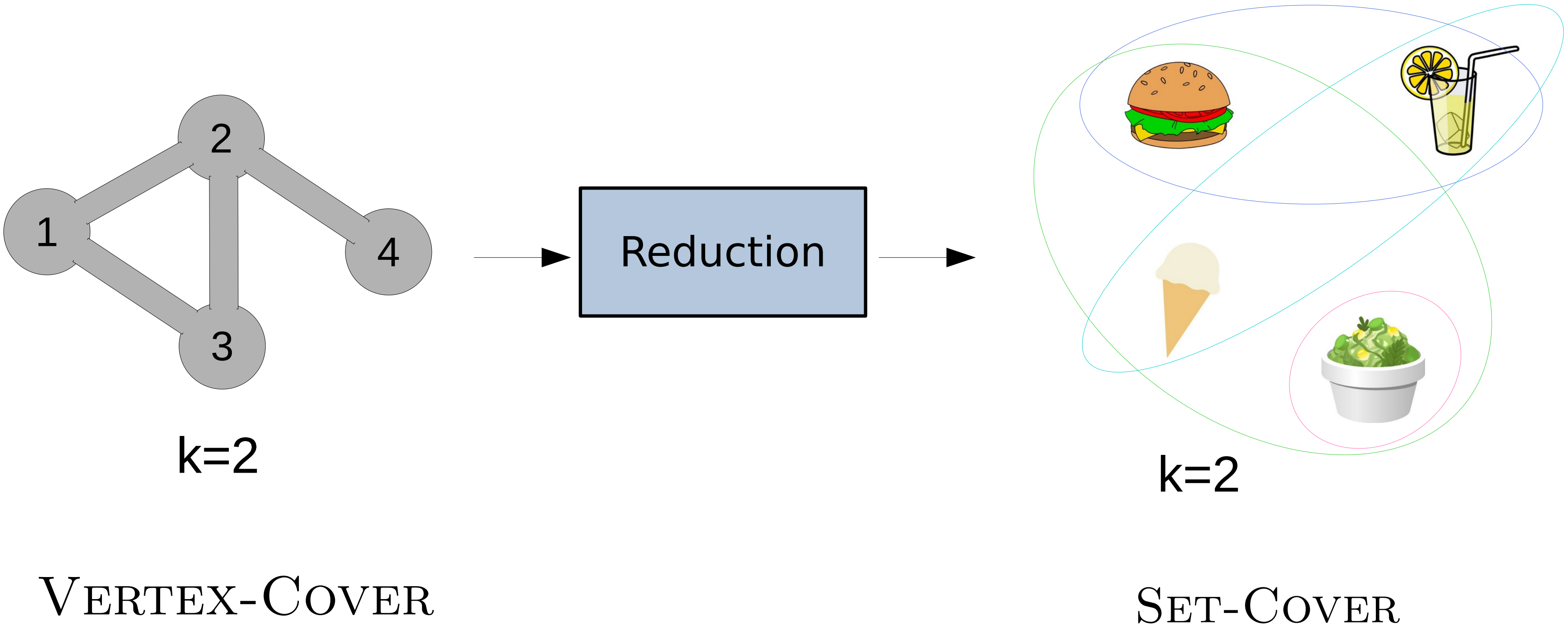


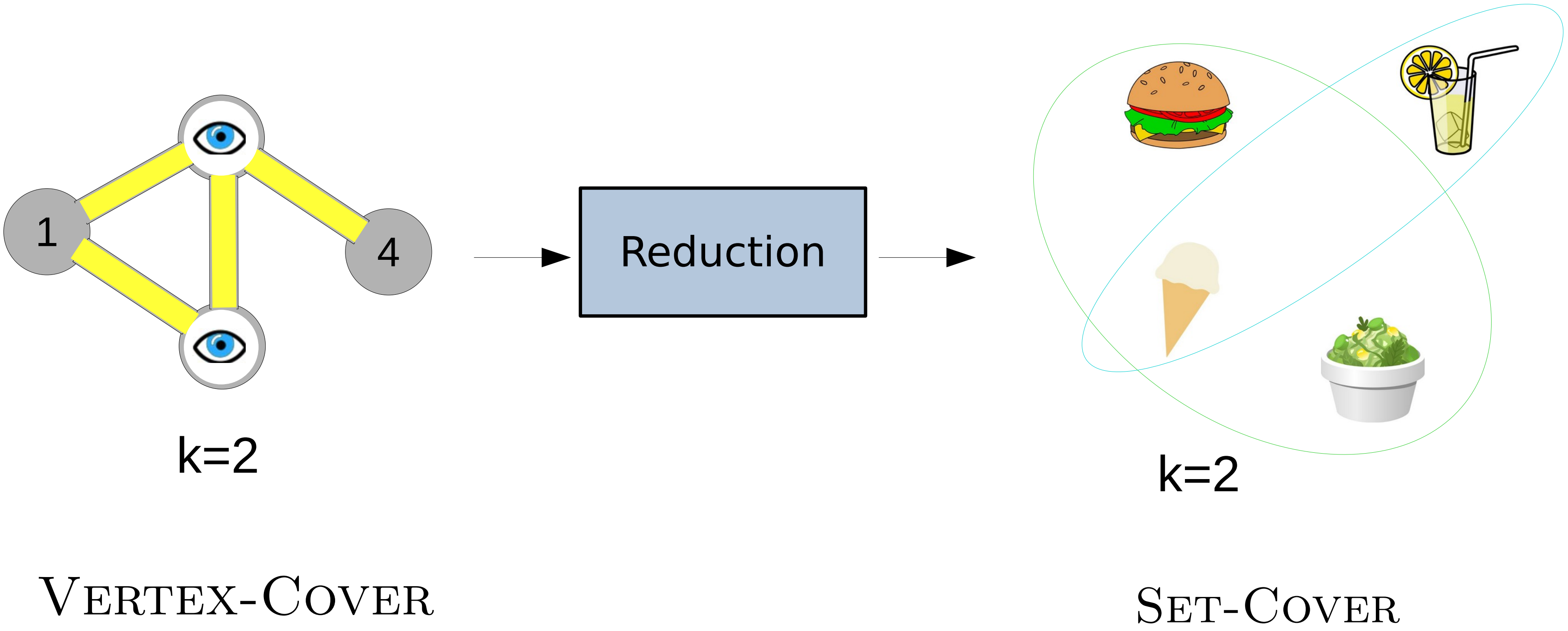
VERTEX-COVER

SET-COVER

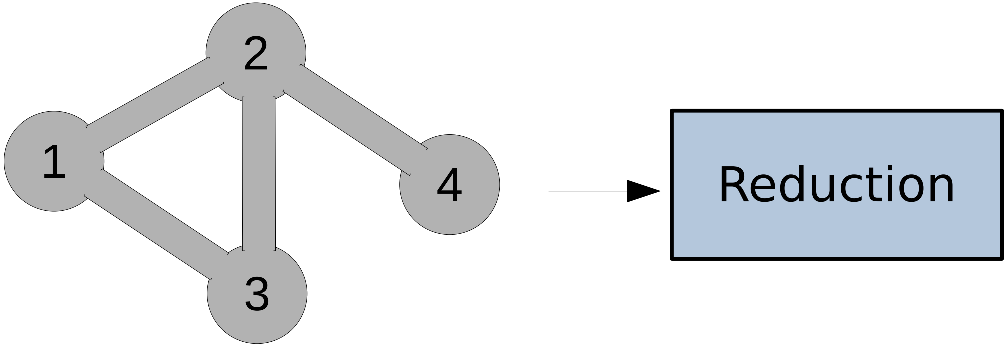
One-Call Reduction



One-Call Reduction



One-Call Reduction

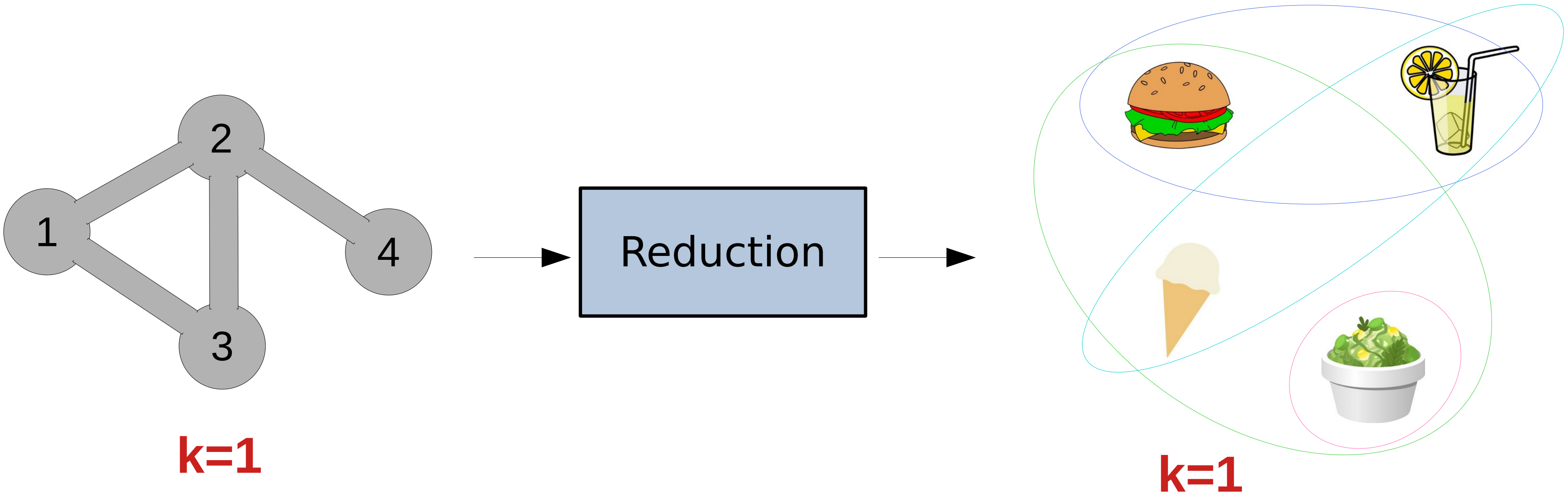


k=1

Vertex-Cover

Set-Cover

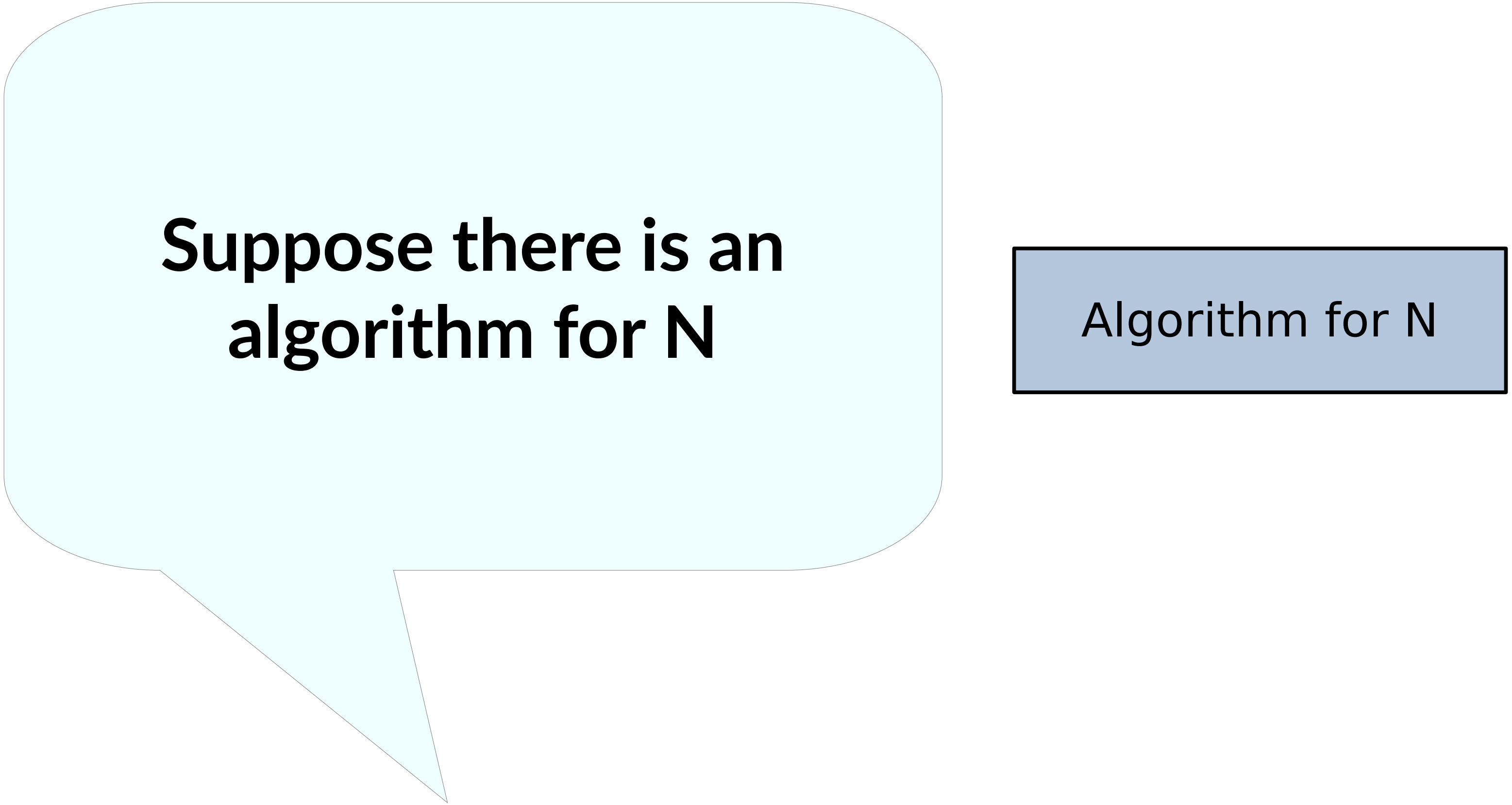
One-Call Reduction



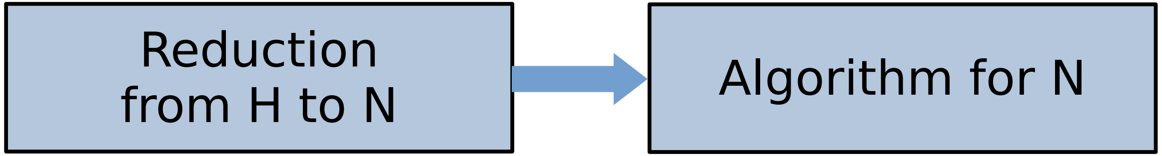
VERTEX-COVER

SET-COVER

One-Call Reduction



One-Call Reduction



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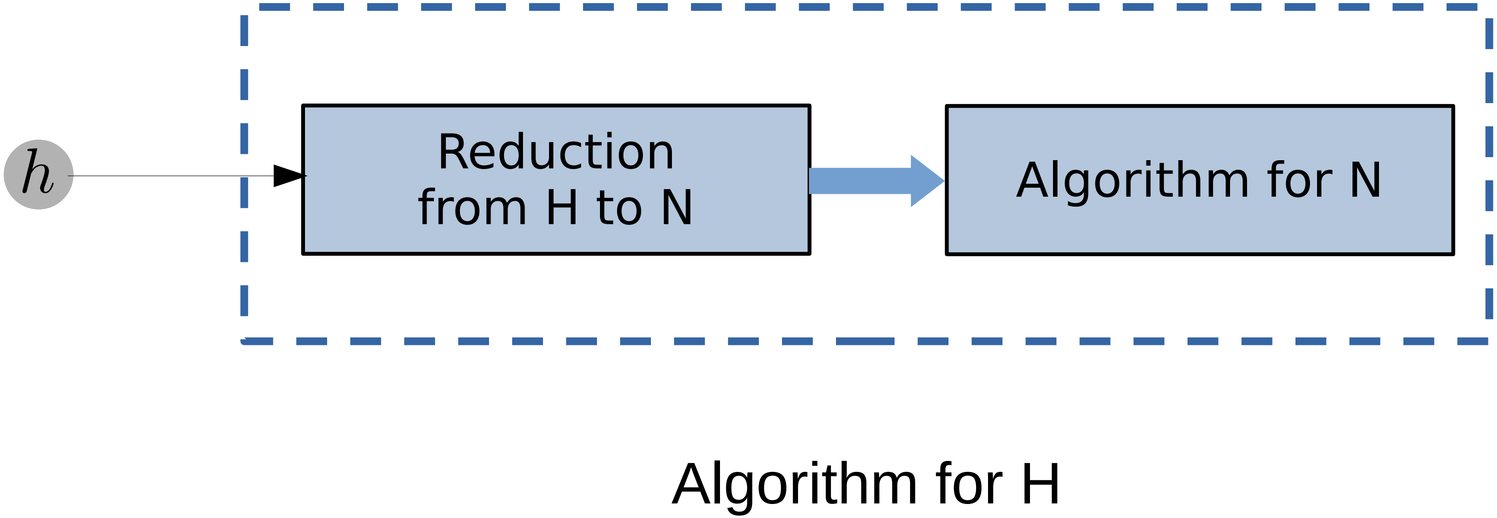

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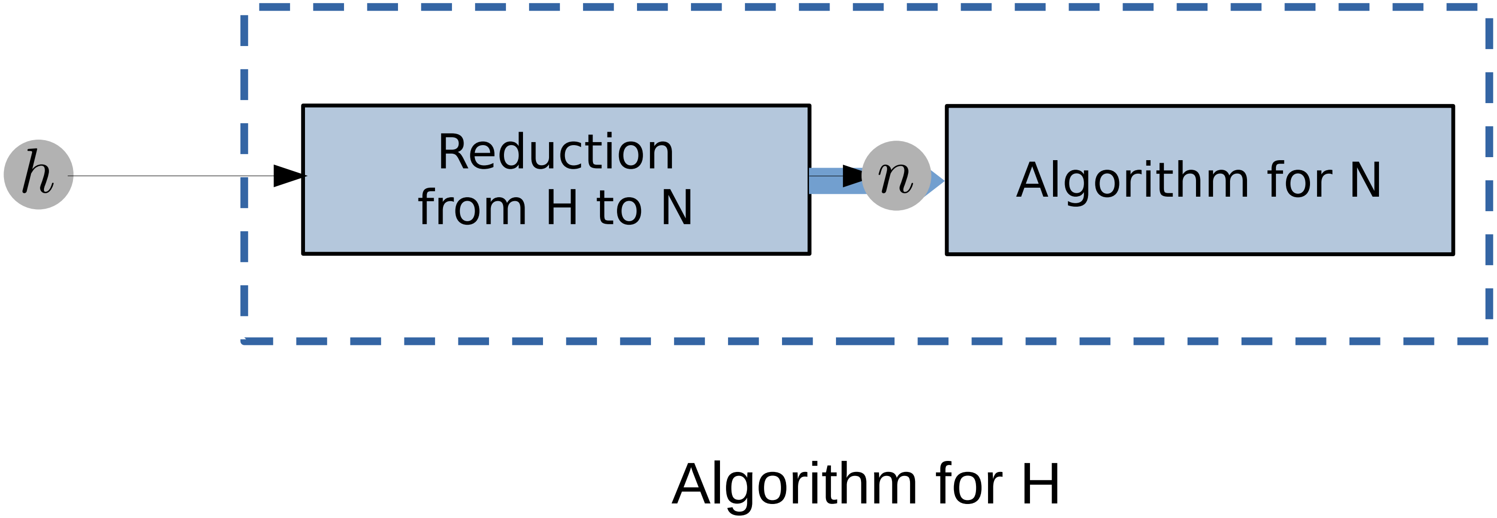

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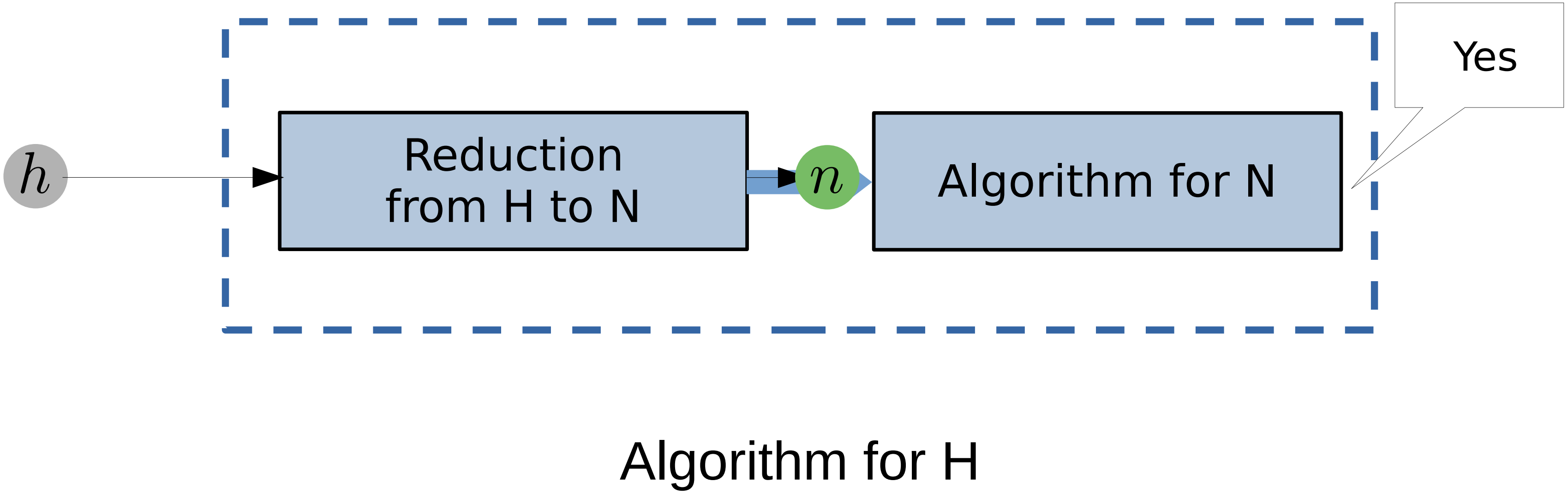

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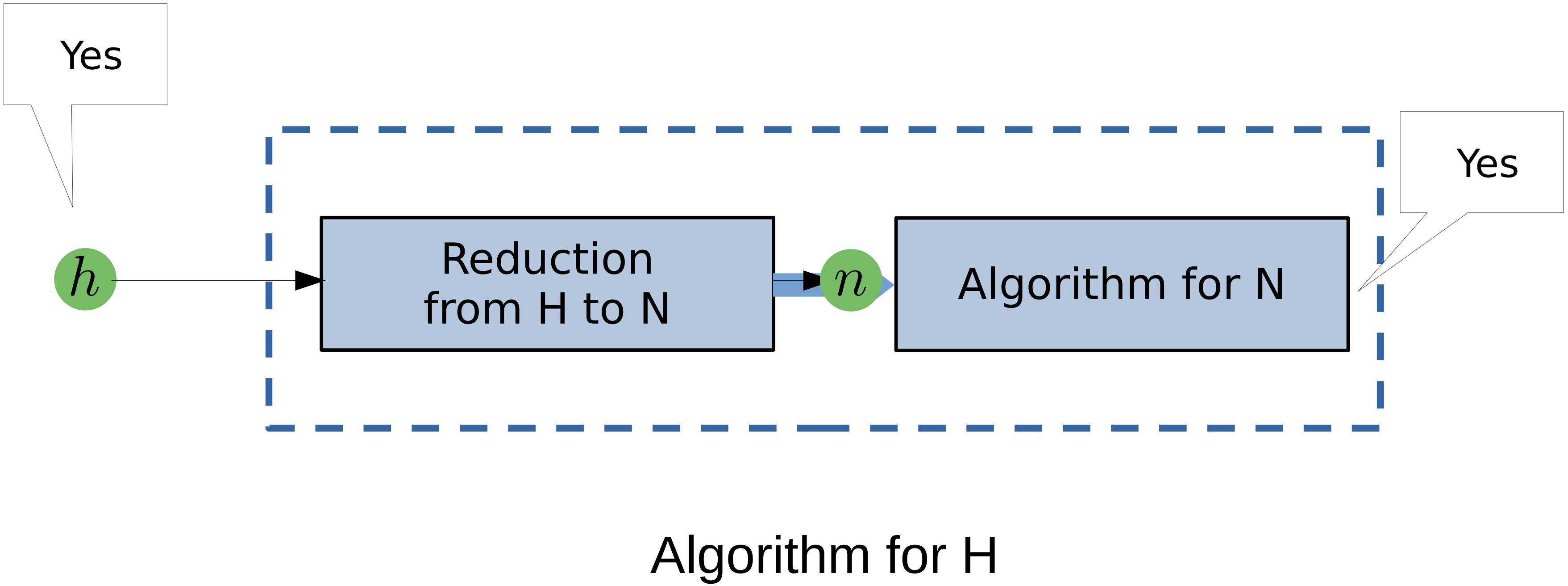

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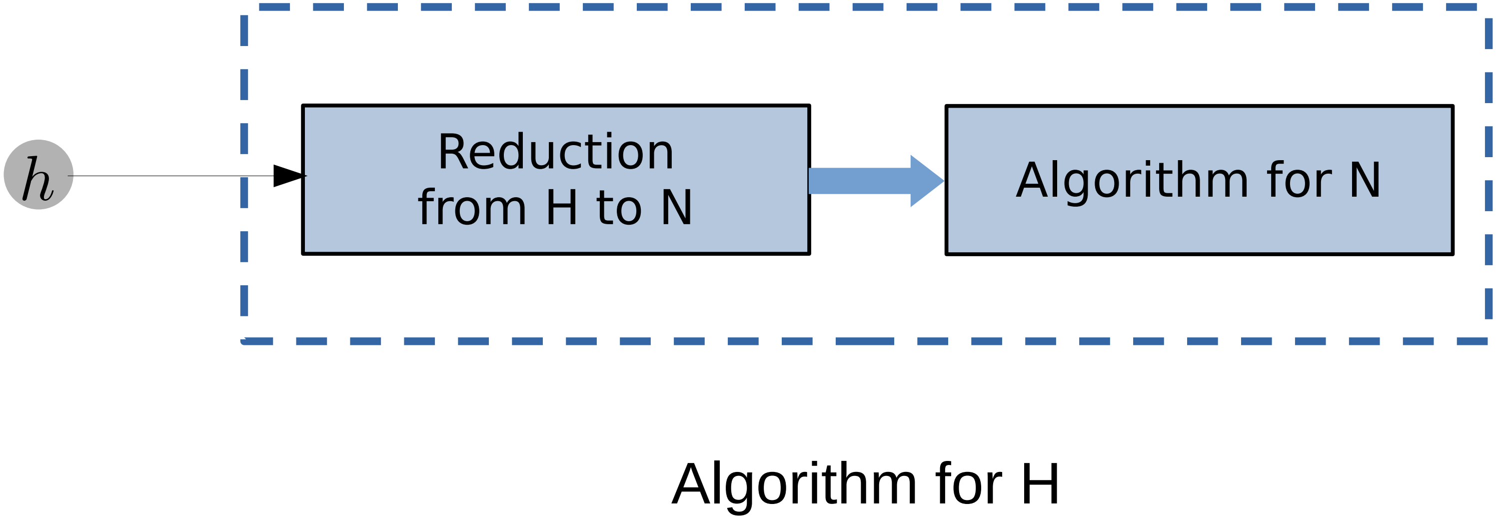

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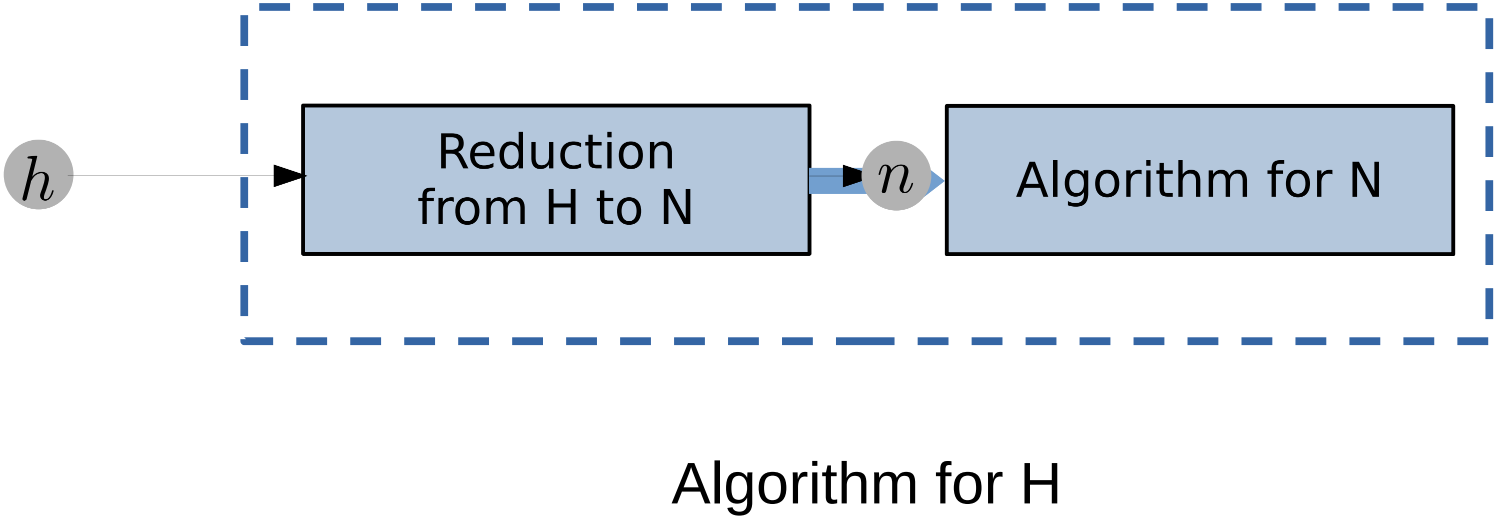
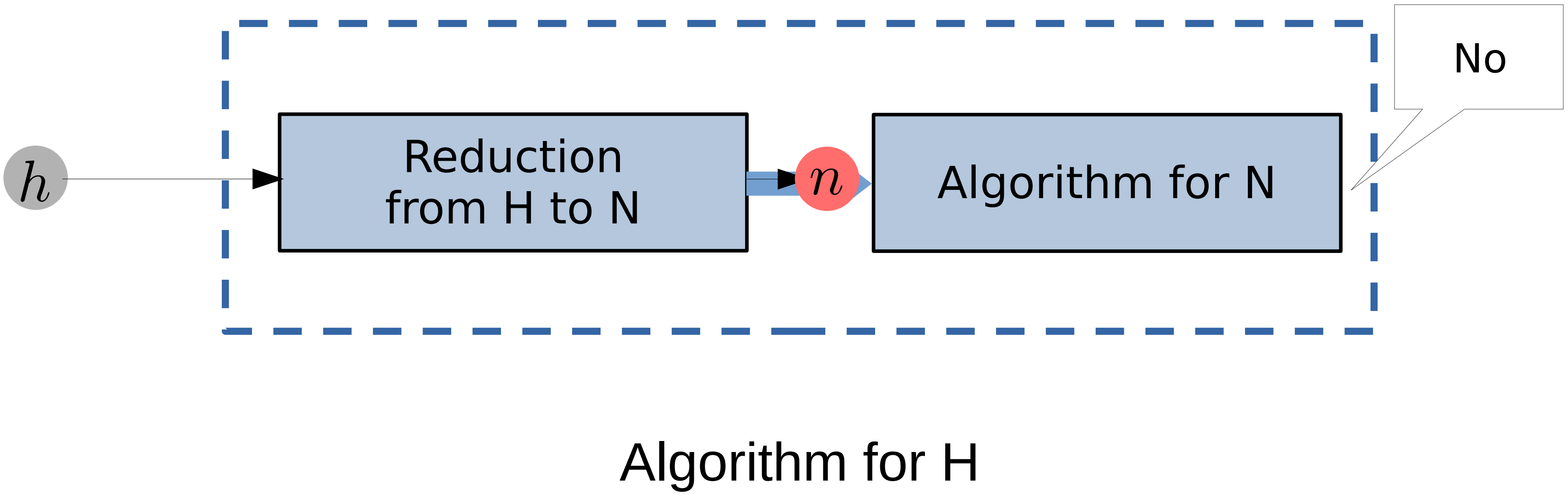
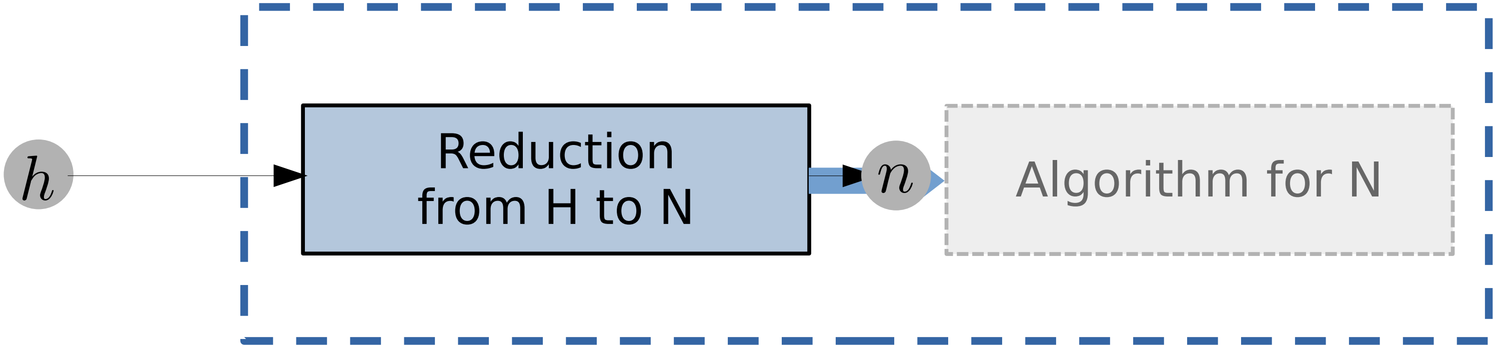


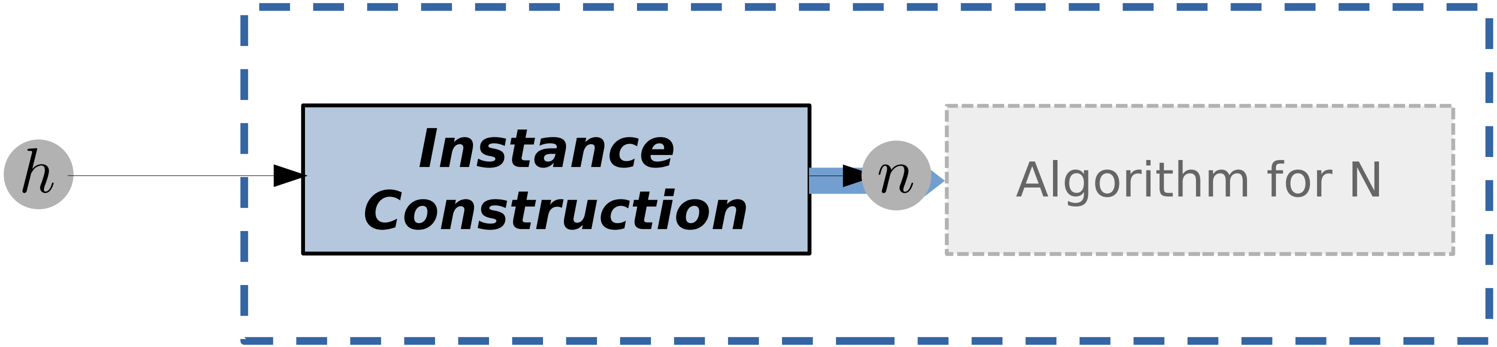
One-Call Reduction



Reduction and Justifications of Correctness



Call this part "*instance construction*" from now on

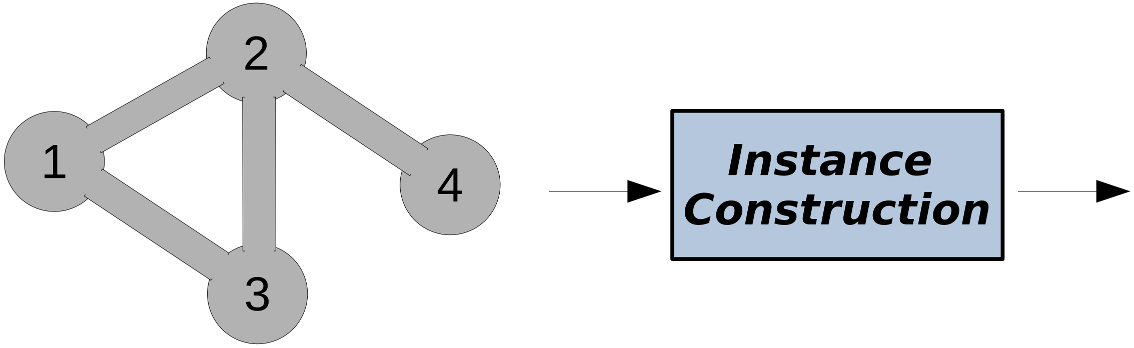


Instance Construction

VERTEX-COVER

SET-COVER

Instance Construction

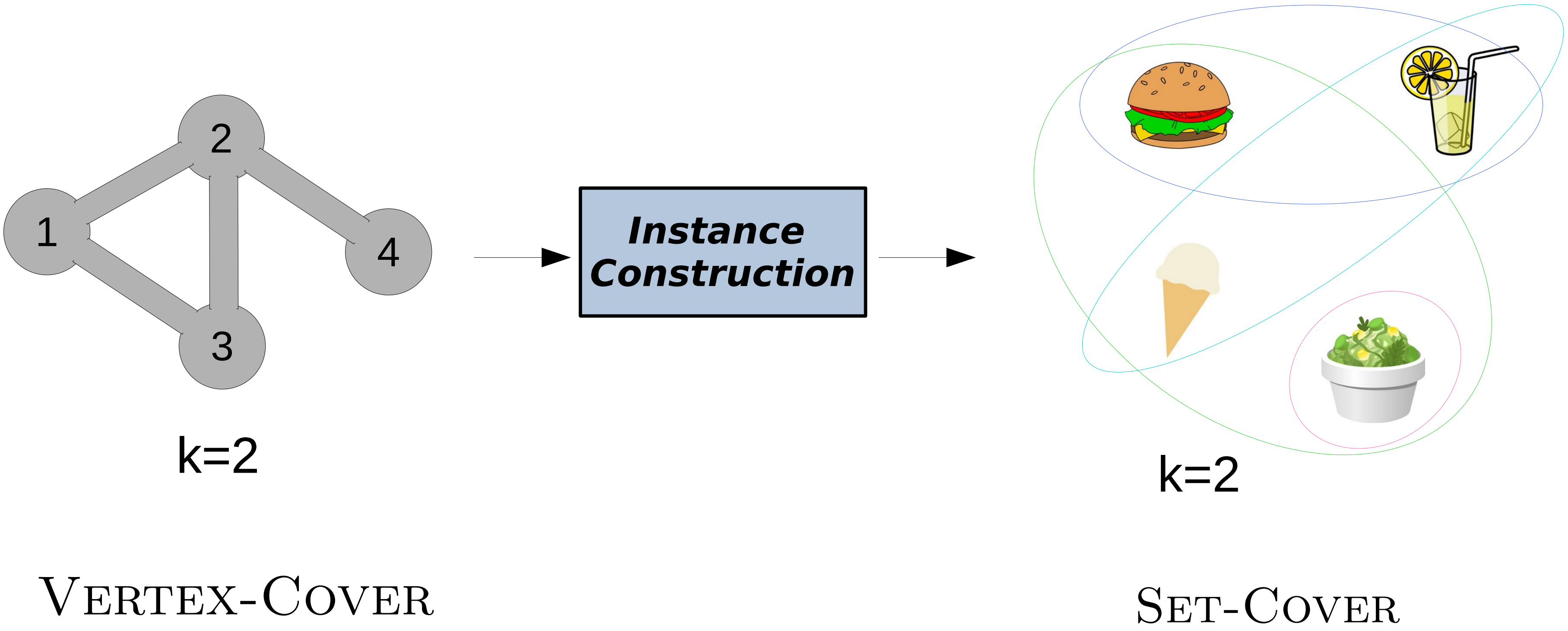


k=2

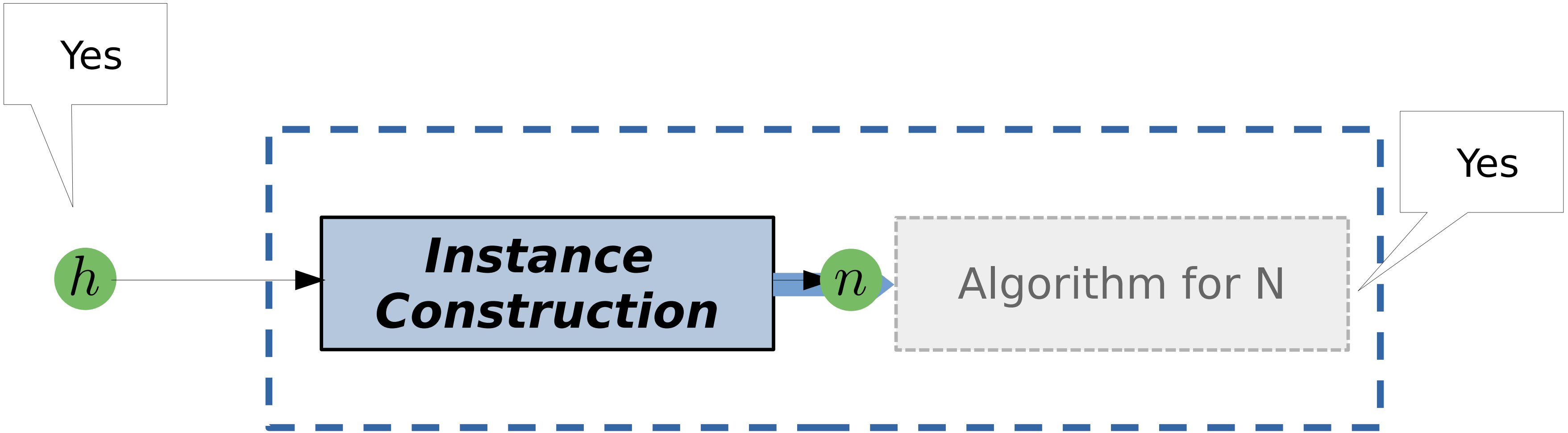
Vertex-Cover

Set-Cover

Instance Construction



Justifying N Yes => H Yes



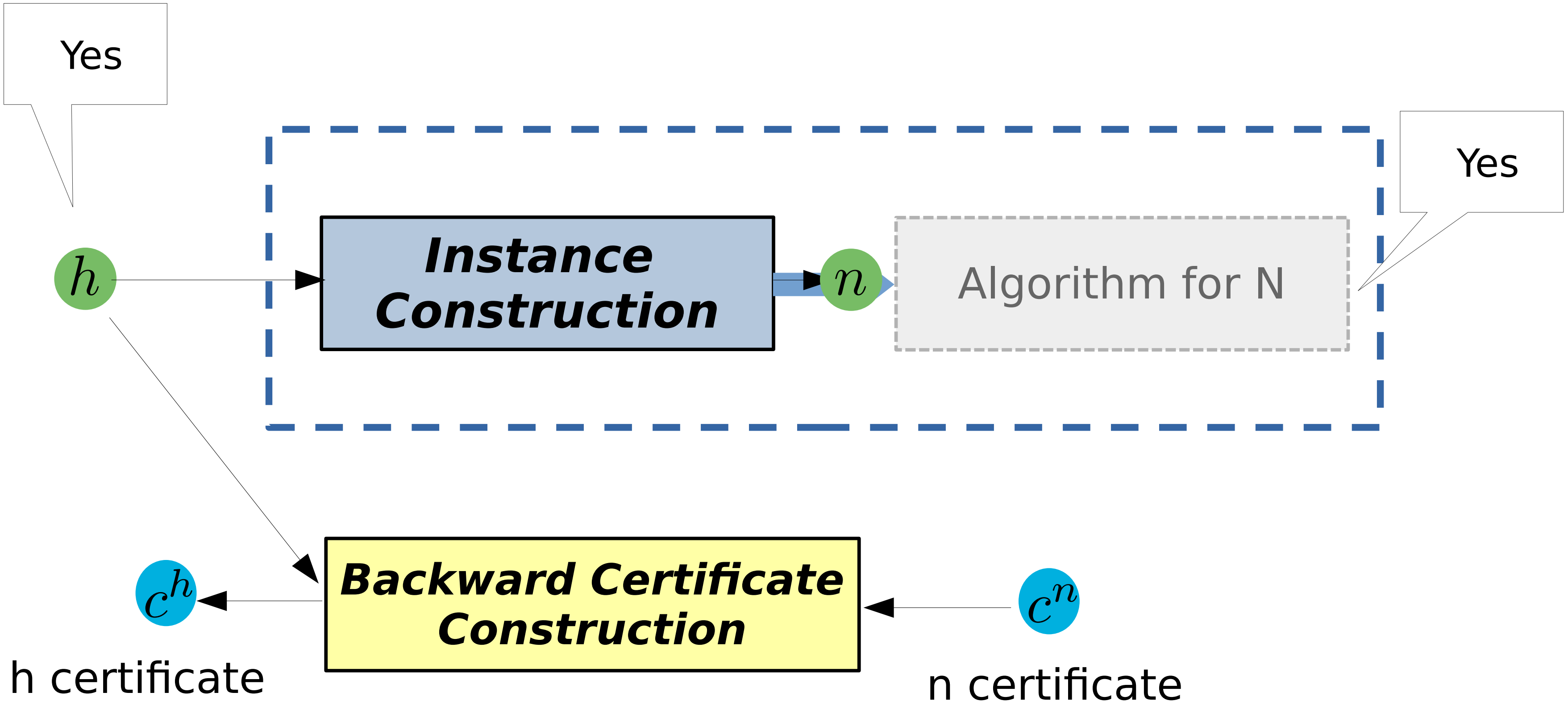
$\exists$ c^h

h certificate

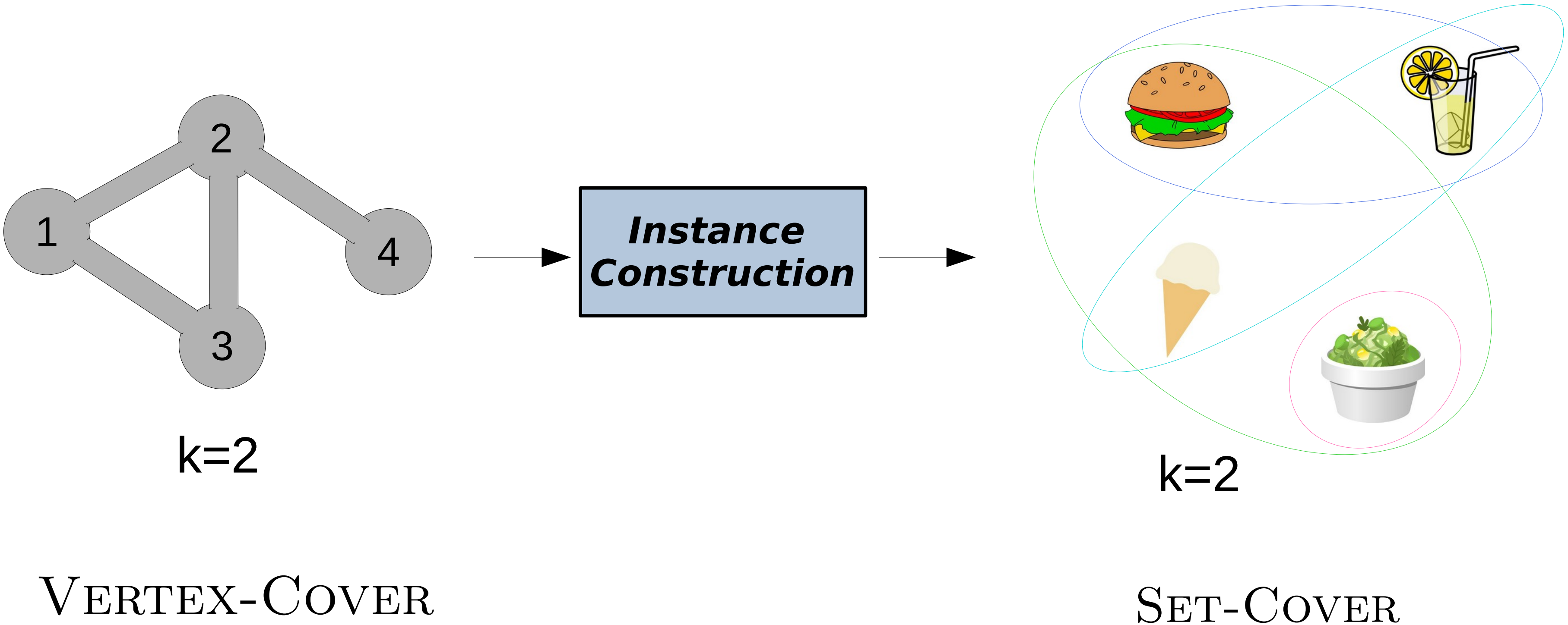
$\exists$ c^n

n certificate

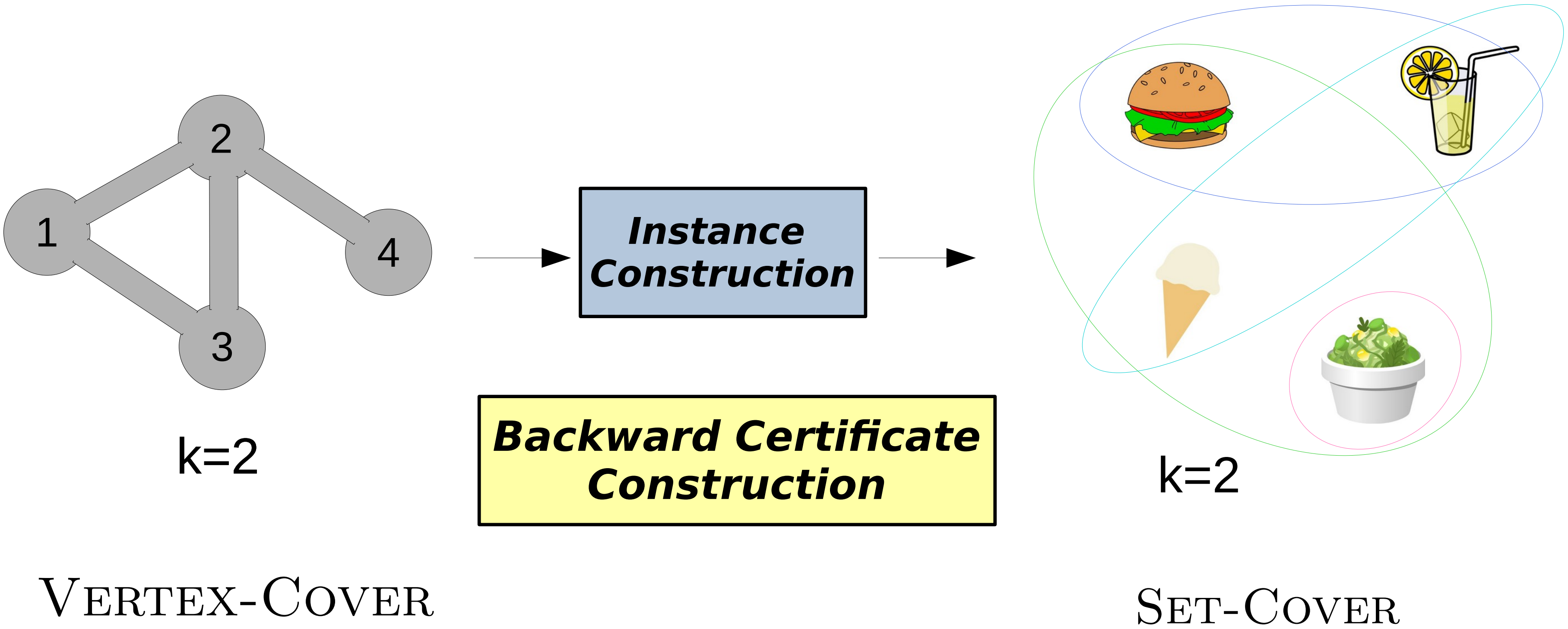
Justifying N Yes => H Yes



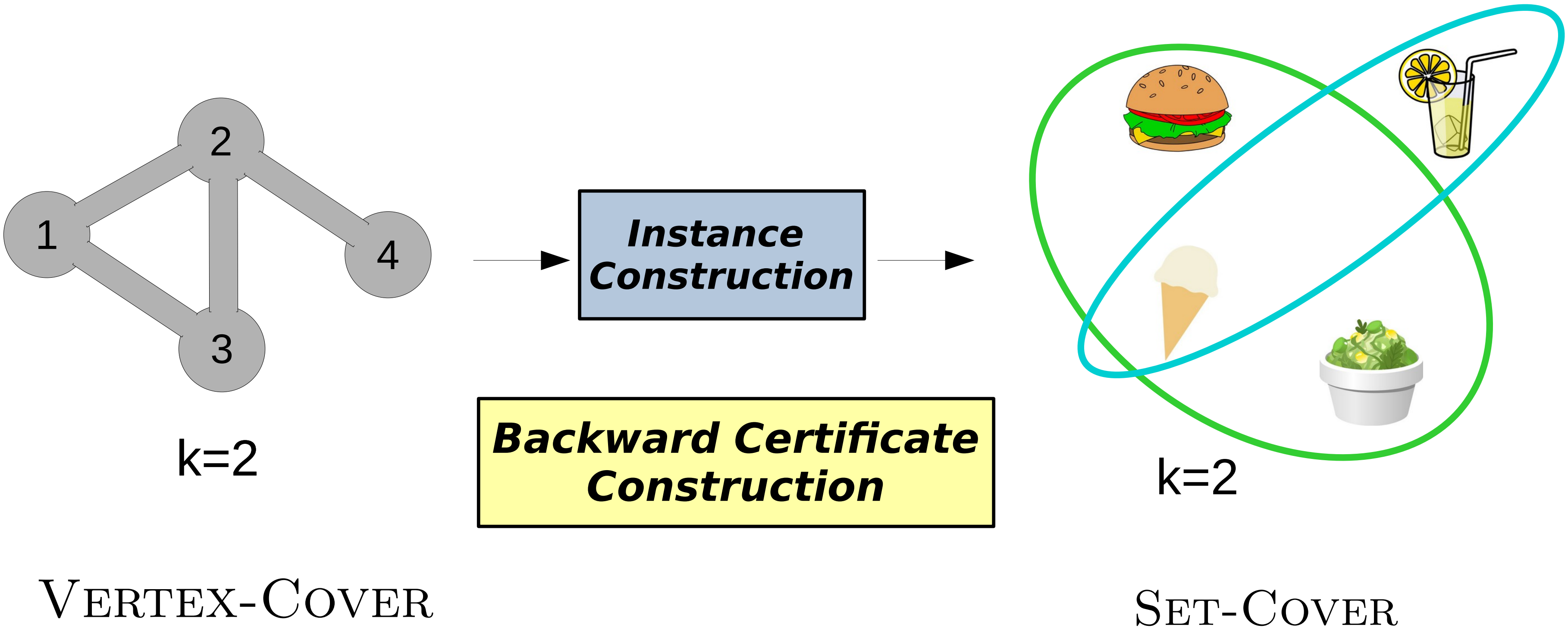
Backward Certificate Construction



Backward Certificate Construction



Backward Certificate Construction



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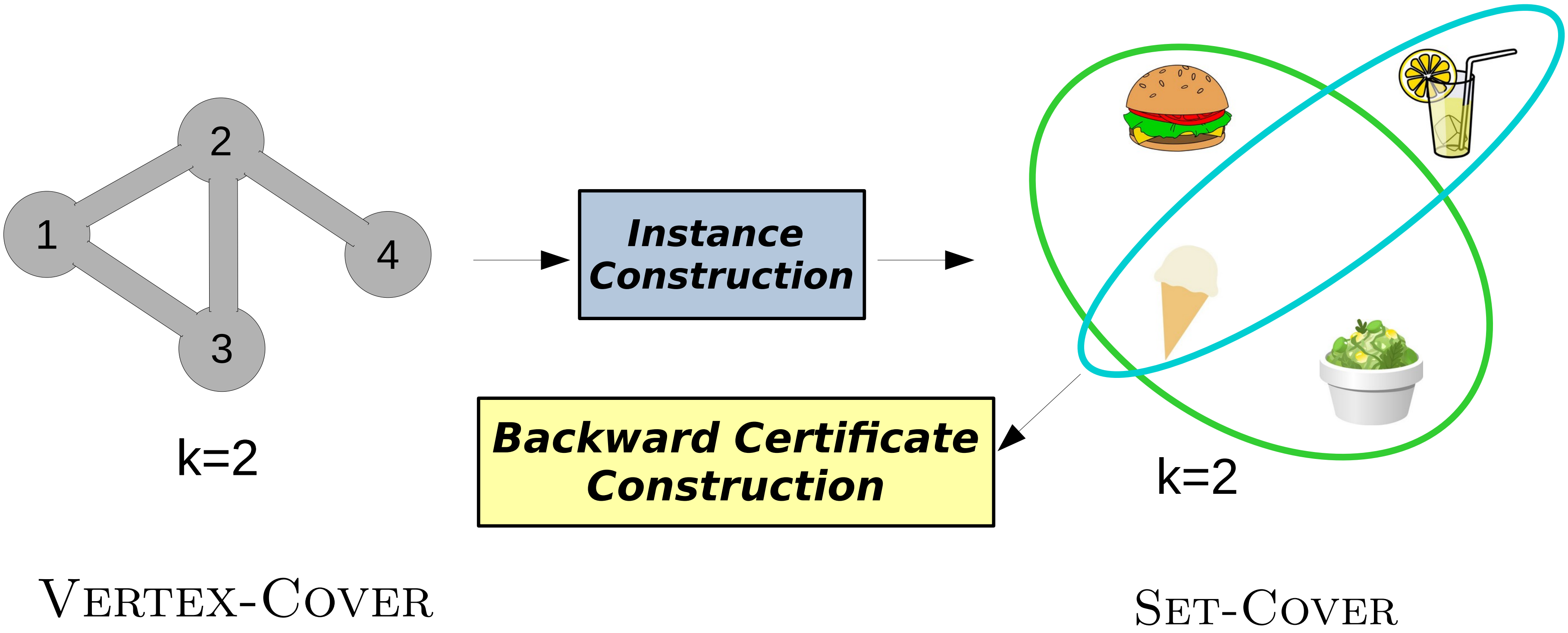

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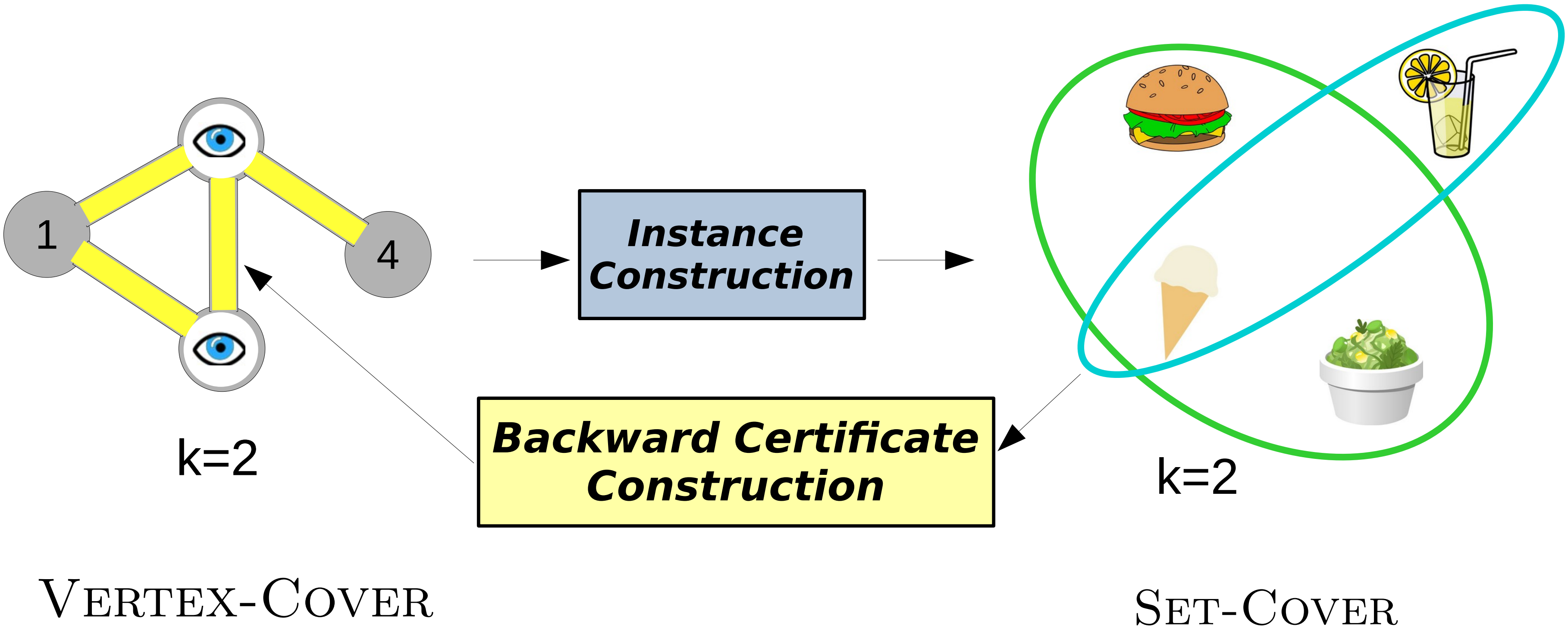
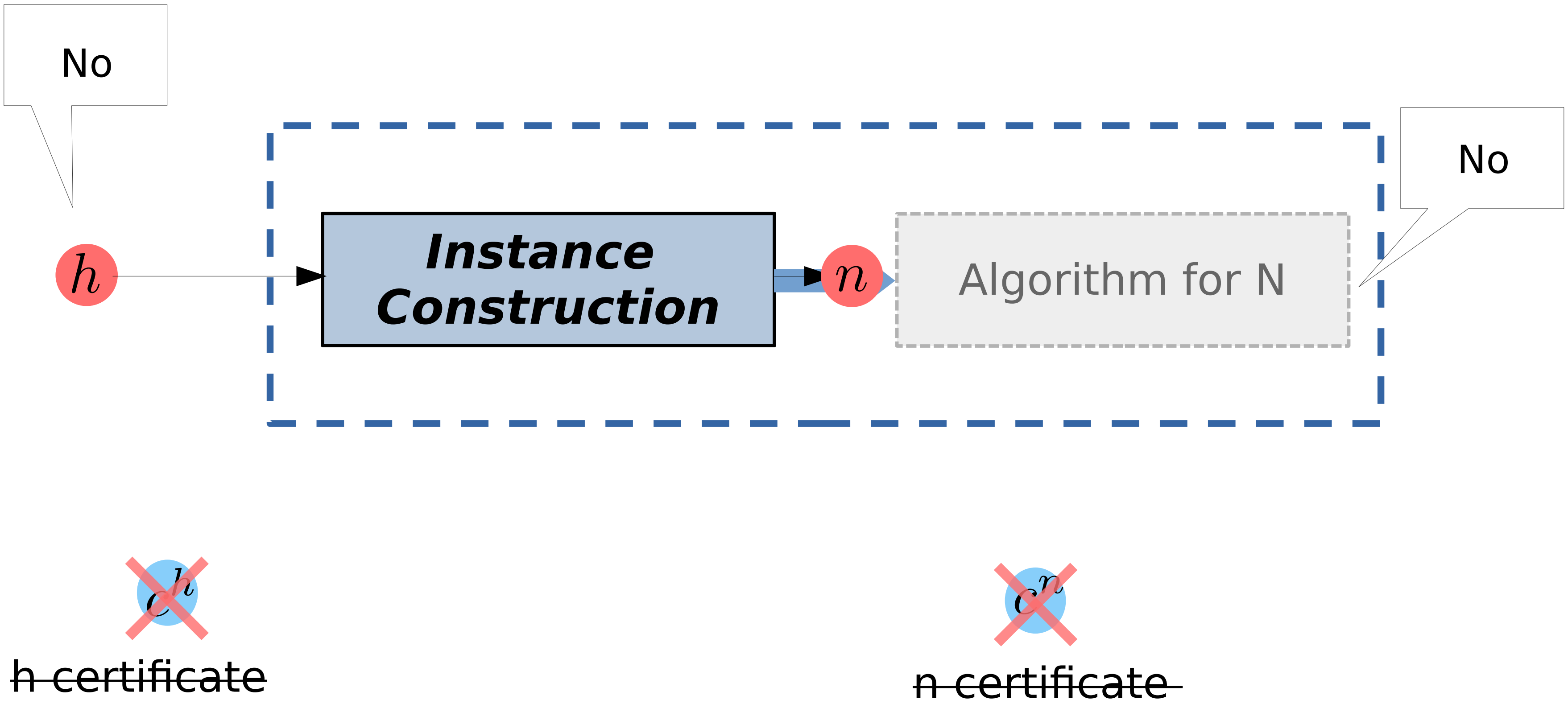
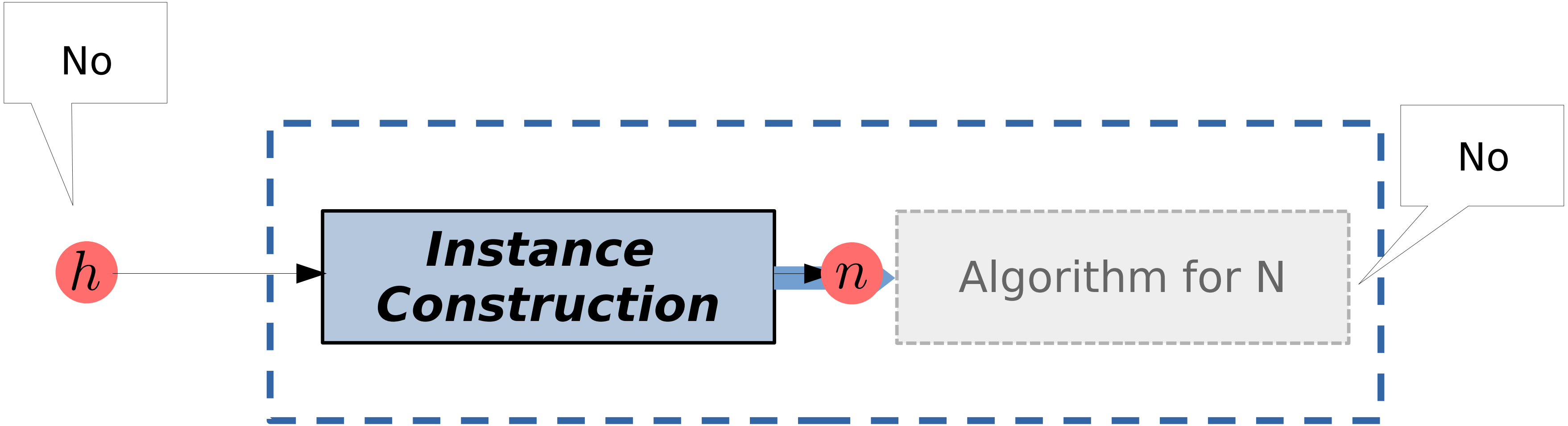


Justifying N No => H No

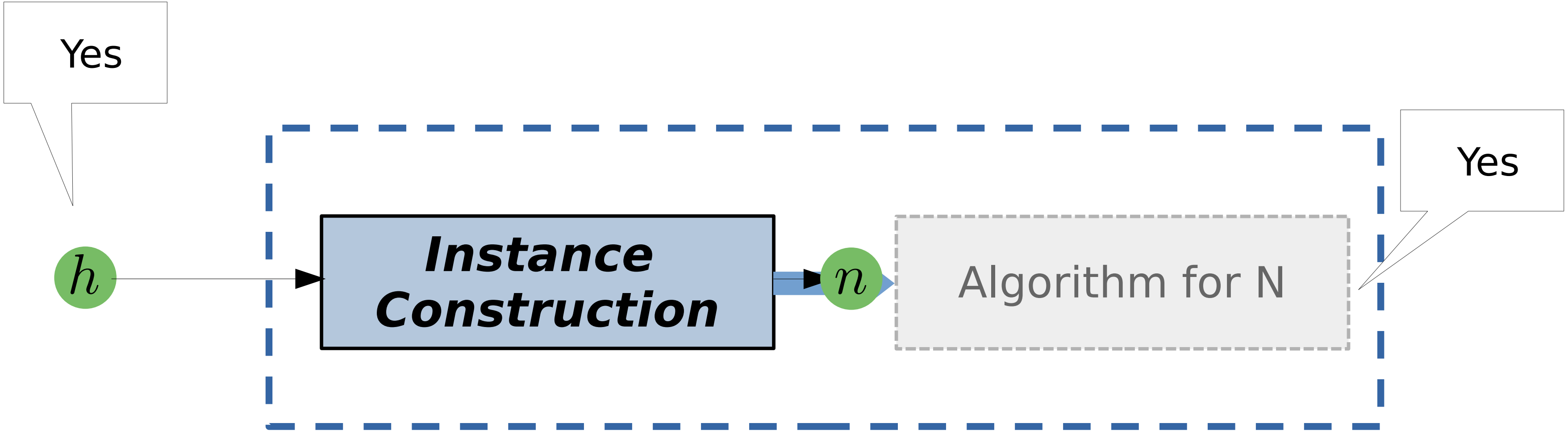


Justifying N No => H No



$$\neg\exists\ c^x \Longleftarrow \neg\exists\ c^y$$

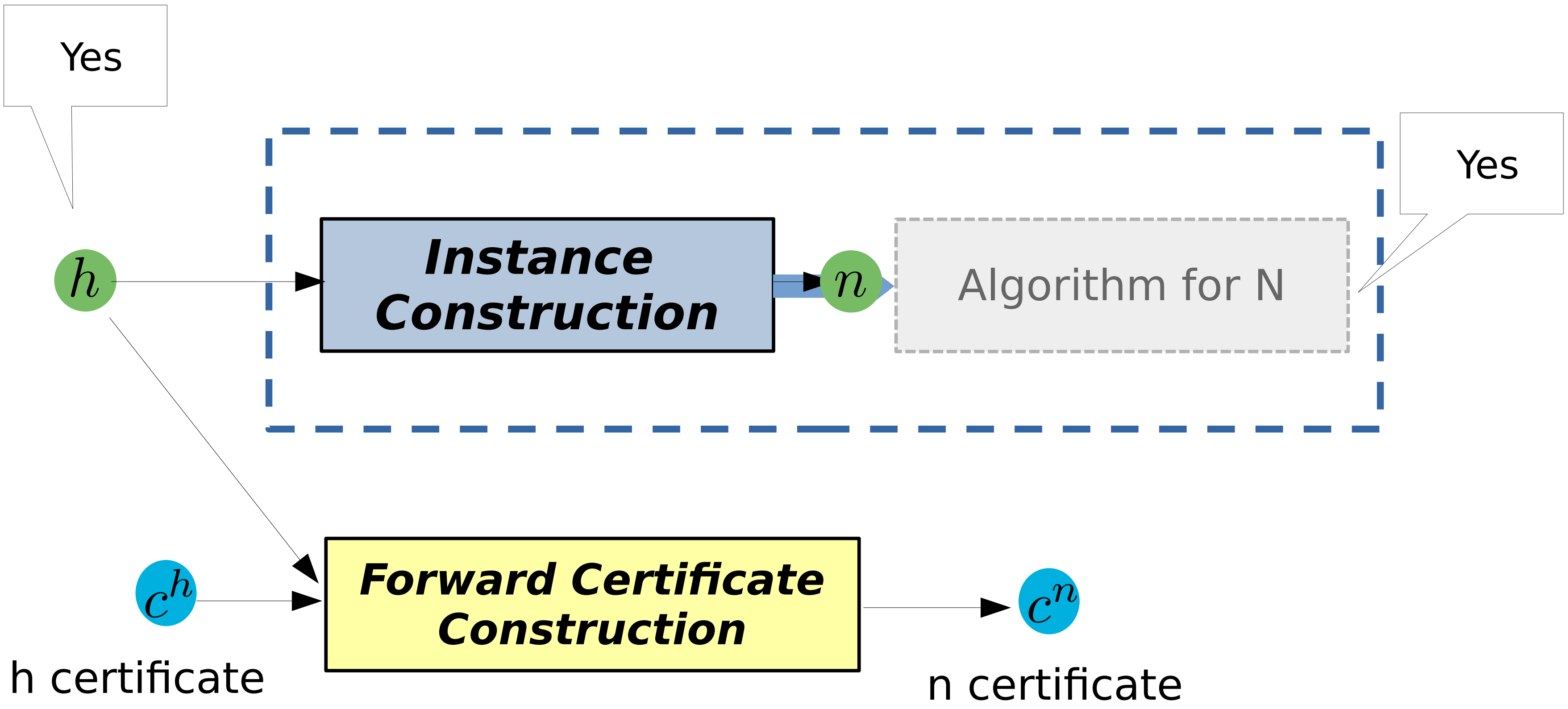
Justifying N No => H No



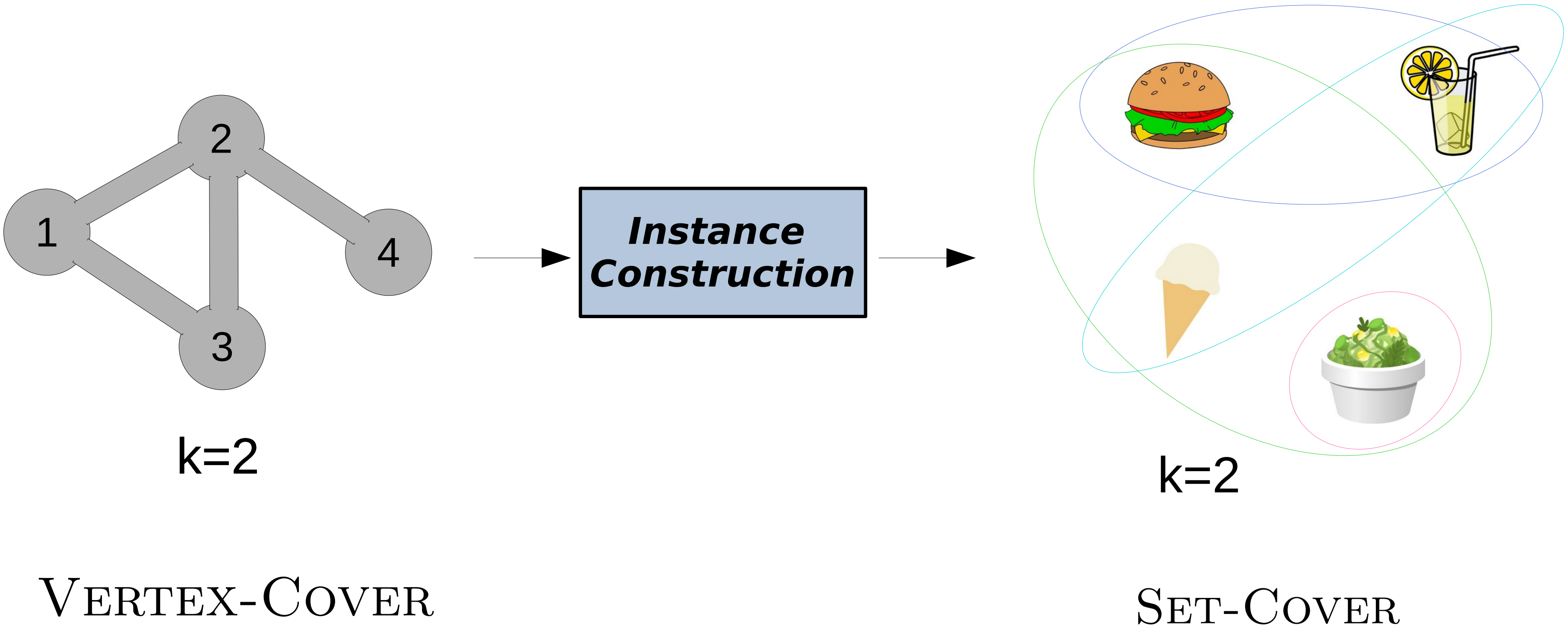
$$\exists\ c^x \implies \exists\ c^y$$

* we are in a classical world

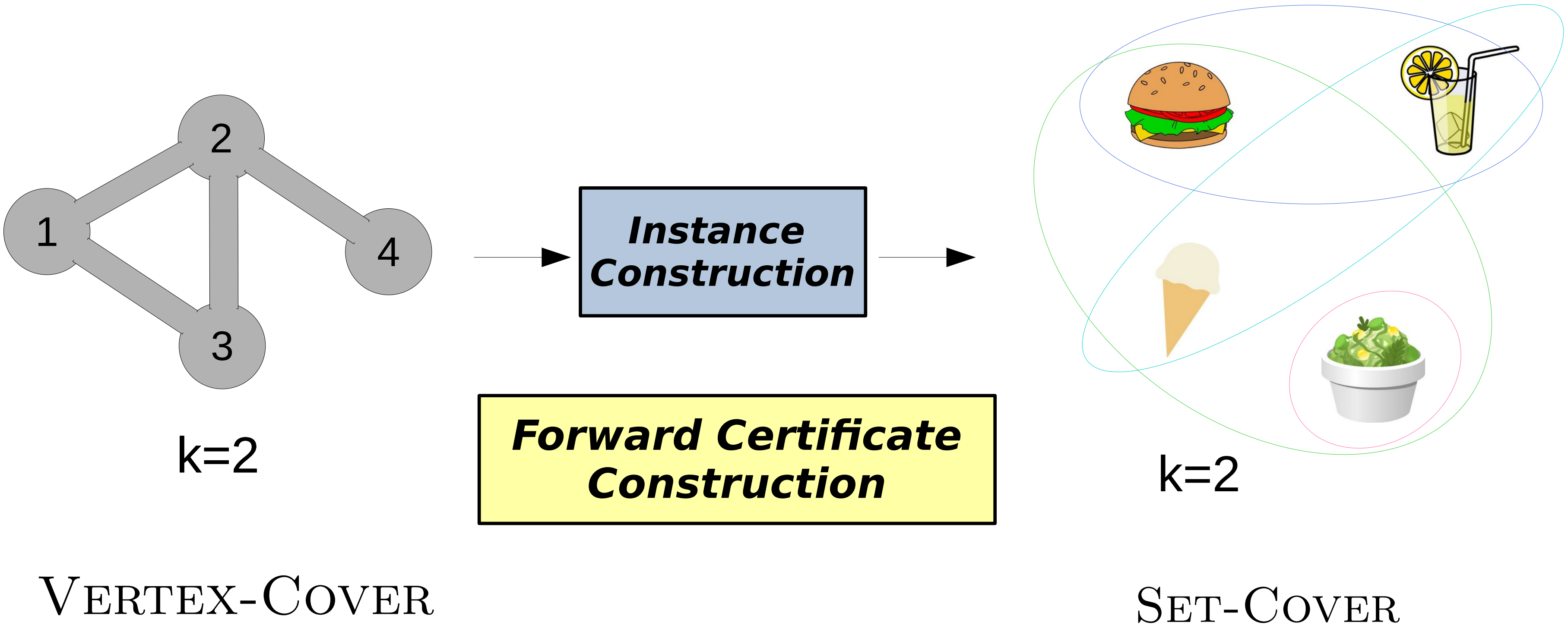
Justifying N No => H No



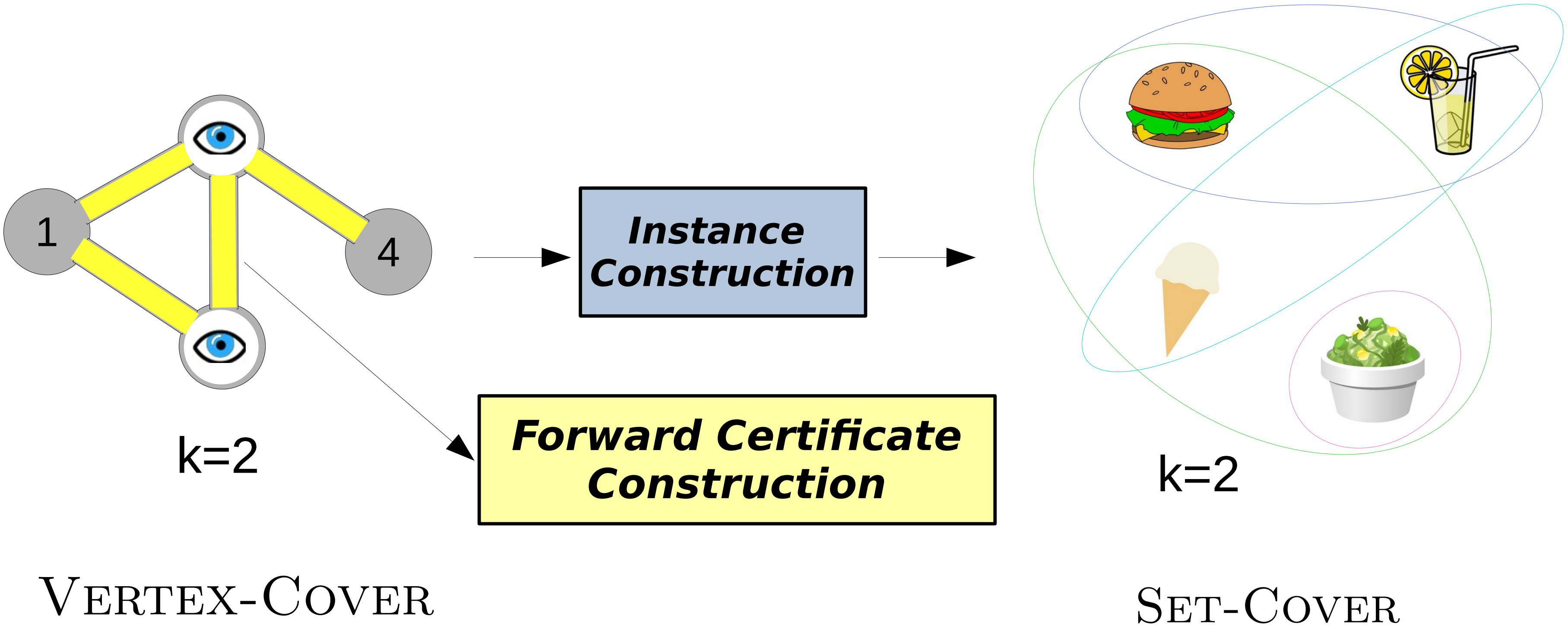
Forward Certificate Construction



Forward Certificate Construction



Forward Certificate Construction



Forward Certificate Construction