Feasible Interior-Point Methods Using Slacks for Nonlinear Optimization

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Background

Interior Methods for solving:

$$\begin{aligned} \min_x \quad & f(x) \\ \text{s.t.} \quad & h(x) = 0 \\ & g(x) \geq 0 \end{aligned}$$

Definition:

Feasible meth. satisfies inequalities, $g(x)$, but not necessarily equalities, at each iteration.

Feasible Methods

- Gay, Overton, Wright (1997)
- Lawrence, Tits (1998)
- Armand, Gilbert, Jan-Jégou (1998)
- Forsgren, Gill (1998)
- ...

Feasible $\Rightarrow$ Infeasible

$$g(x) \geq 0 \quad \Rightarrow \quad g(x) - s = 0, \; s \geq 0$$

Infeasible Methods

Infeasible $\Rightarrow$ Feasible

- General framework (LS or TR)
- Incorporates slacks (uniform structure)

Infeasible framework

$$\begin{aligned} \min_{x,s} \quad & f(x) - \mu \sum_{i \in \mathcal{I}} \ln(s_i) \\ \text{s.t.} \quad & h(x) = 0 \\ & g(x) - s = 0 \\ & s > 0 \end{aligned}$$

Infeasible Methods

- Vanderbei, Shanno [LOQO] (2000)
- Toint (1998)
- Yamashita, Yabe, Tanabe (1997)
- Byrd, Hribar, Nocedal [NITRO] (1999)
- ...

Feasible Method Using Slacks

Assume merit function, ϕ, has the form

$$\phi(x,s) \quad = \quad f(x) - \mu \sum_{i \in \mathcal{I}} \ln(s_i) + \chi(h(x), g(x,s))$$
$$\phi(x,s) \quad \to \quad \infty \quad \text{as} \quad s \to 0$$

Feasible Algorithm Step Computation

1. Assume current point (x_k, s_k) is feasible.
2. Set $s_k = g(x_k)$
3. After taking a trial step, $x_\mathsf{T} = x_k + d_x$ redefine the slacks as

$$\boxed{s_\mathsf{T} \leftarrow g(x_\mathsf{T})}$$

4. While ϕ does not decrease sufficiently

   Compute a shorter step d_x

   Set $x_\mathsf{T} = x_k + d_x$; $\boxed{s_\mathsf{T} = g(x_\mathsf{T})}$

   End While

Feasible Method Using Slacks

Line Search methods:

- everything fine

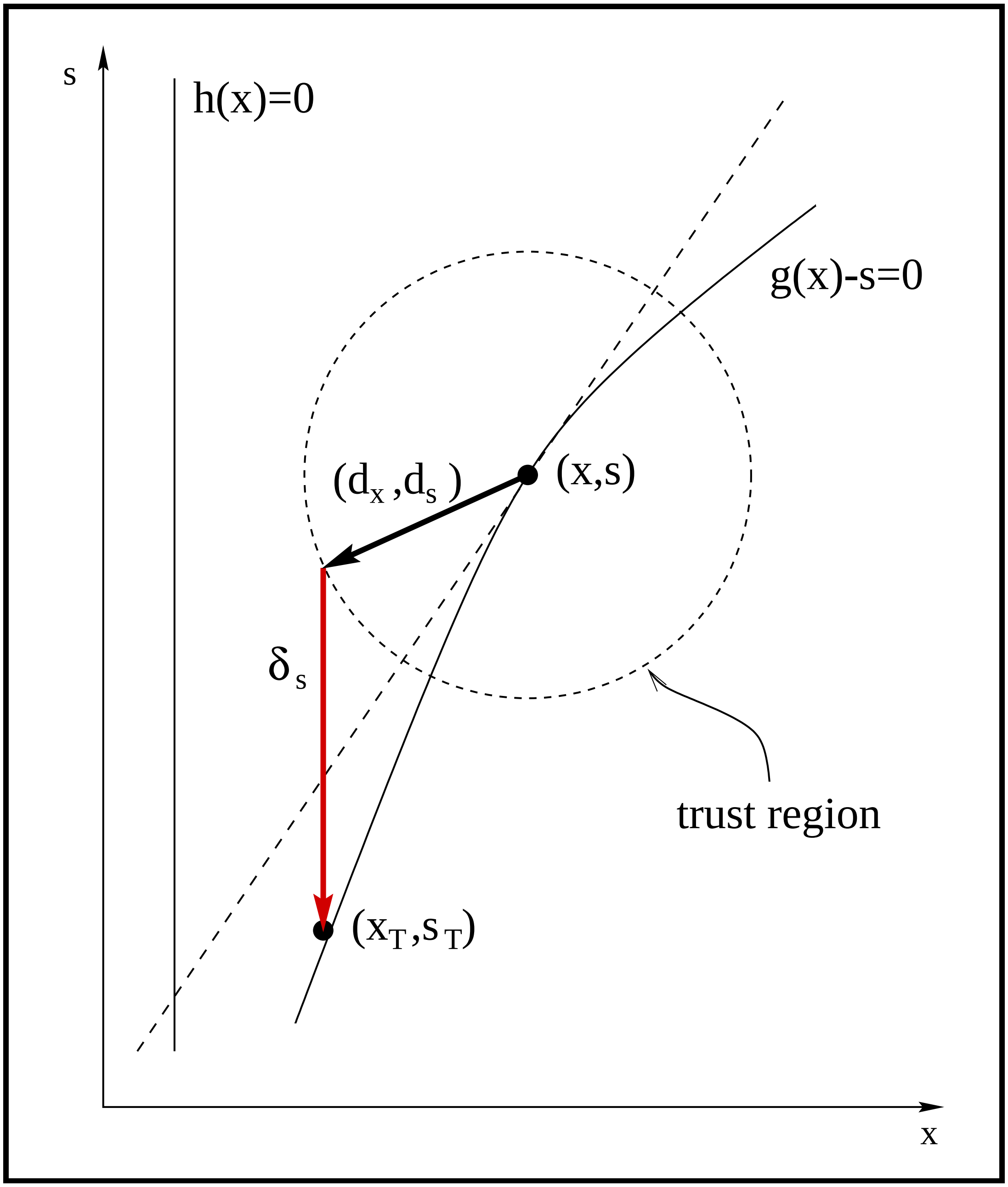
Trust Region methods:

- everything usually fine
- potential problem if Cauchy component active

Trust Region Reset Difficulty

Problem: Slack reset may increase merit function!

$$s_\top = g(x_\top) = s + d_s + \delta_s$$



No Equalities

- Assume there are **no** equalities
- From the slack reset, $g(x) - s = 0$
- The step (d_x, d_s) will maintain feasibility with respect to the linearization of $g(x) - s = 0$

$$\|g(x + d_x) - (s + d_s)\| = O(\|d_x\|^2).$$

- We know from the slack reset that

$$s_T = g(x + d_x) = s + d_s + \delta_s$$

Therefore

$$\boxed{\|\delta_s\| = O(\|d_x\|^2)}$$

- Merit function, ϕ, will decrease for small d_x

With Equalities

1. Line Search

   - Step direction satisfies linearized constraints

   - No problem: $\|\delta_s\| = O(\|d_x\|^2)$

2. Trust Region

   - Trust region constraint and satisfaction of linearized constraints may be incompatible

$$\|g(x + d_x) - (s + d_s)\| = O(\|d_x\|)$$

$$\boxed{\|\delta_s\| = O(\|d_x\|)}$$

   - **No guarantee merit function, $\phi(x, s)$, will decrease for small d_x!**

Solution

Ensure step (d_x, d_s) satisfies 2 conditions:

1. Satisfy linearization of $g(x) - s$ always

$$A_g(x)^T d_x - d_s = 0$$

$$A_g(x) = [\nabla g_1(x), \nabla g_2(x), ..., \nabla g_m(x)]$$

2. Make sufficient progress towards feasibility of violated equality constraints.

- Solution is algorithm dependent

NITRO solution

$$d = n + t$$

- $n = (n_x, n_s) =$ "Normal (feasibility) step"
- Normal step subproblem:

$$\begin{array}{ll} \min & \|A_{\mathsf{h}}^T n_x + h\|_2^2 + \|A_{\mathsf{g}}^T n_x + g - n_s - s\|_2^2 \\ \text{s.t.} & \|(n_x, n_s)\|_2 \le \Delta \\ & s + n_s > 0 \end{array}$$

- Newton step satisfies the linear. constraints
- Cauchy direction may violate linearized constraints

$$\begin{aligned} n_x^C &= -A_{\mathsf{h}} h(x) + A_{\mathsf{g}} \overbrace{(g(x) - s)}^{0} \\ &= -A_{\mathsf{h}} h(x) \end{aligned}$$

- Need a new Cauchy-type step (n_x^C, n_s^C)
- Solve via Dogleg method.

Modified Cauchy Step

Conditions:

1. $(n_x^C, n_s^C) \in \text{range}(A^T)$

2. (n_x^C, n_s^C) satisfy linearization of $g(x) - s = 0$

$$A_{\mathsf{g}}^T n_x^C - n_s^C = 0$$

3. n_x^C provide a reduction in the linearized equality constraints.

   Normal Cauchy step:

$$n_x^C = -A_{\mathsf{h}} h(x)$$

   Modified Cauchy step

$$\boxed{A_{\mathsf{h}}^T n_x^C = -A_{\mathsf{h}}^T A_{\mathsf{h}} h}$$

$$\begin{aligned} \|h + \alpha A_{\mathsf{h}}^T n_x^C\|^2 &= \|h + \alpha A_{\mathsf{h}}^T A_{\mathsf{h}} h\|^2 \\ &= \|h\|^2 - 2\alpha \|A_{\mathsf{h}} h\|^2 + O(\alpha^2) \end{aligned}$$

Modified Cauchy Step

- *Modified Cauchy step* is the solution of

$$\begin{bmatrix} I & 0 & A_{\mathsf{h}} & A_{\mathsf{g}} \\ 0 & I & 0 & -I \\ A_{\mathsf{h}}^T & 0 & 0 & 0 \\ A_{\mathsf{g}}^T & -I & 0 & 0 \end{bmatrix} \begin{bmatrix} n_x^C \\ n_s^C \\ p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -A_{\mathsf{h}}^T A_{\mathsf{h}} h \\ 0 \end{bmatrix}$$

- Cauchy step - no matrix inversion

- Coefficient matrix already exists (tangential step, Lagrange multipliers)

- Extra cost = one backsolve

Equivalence of Slack-Based and Classical Feasible Methods

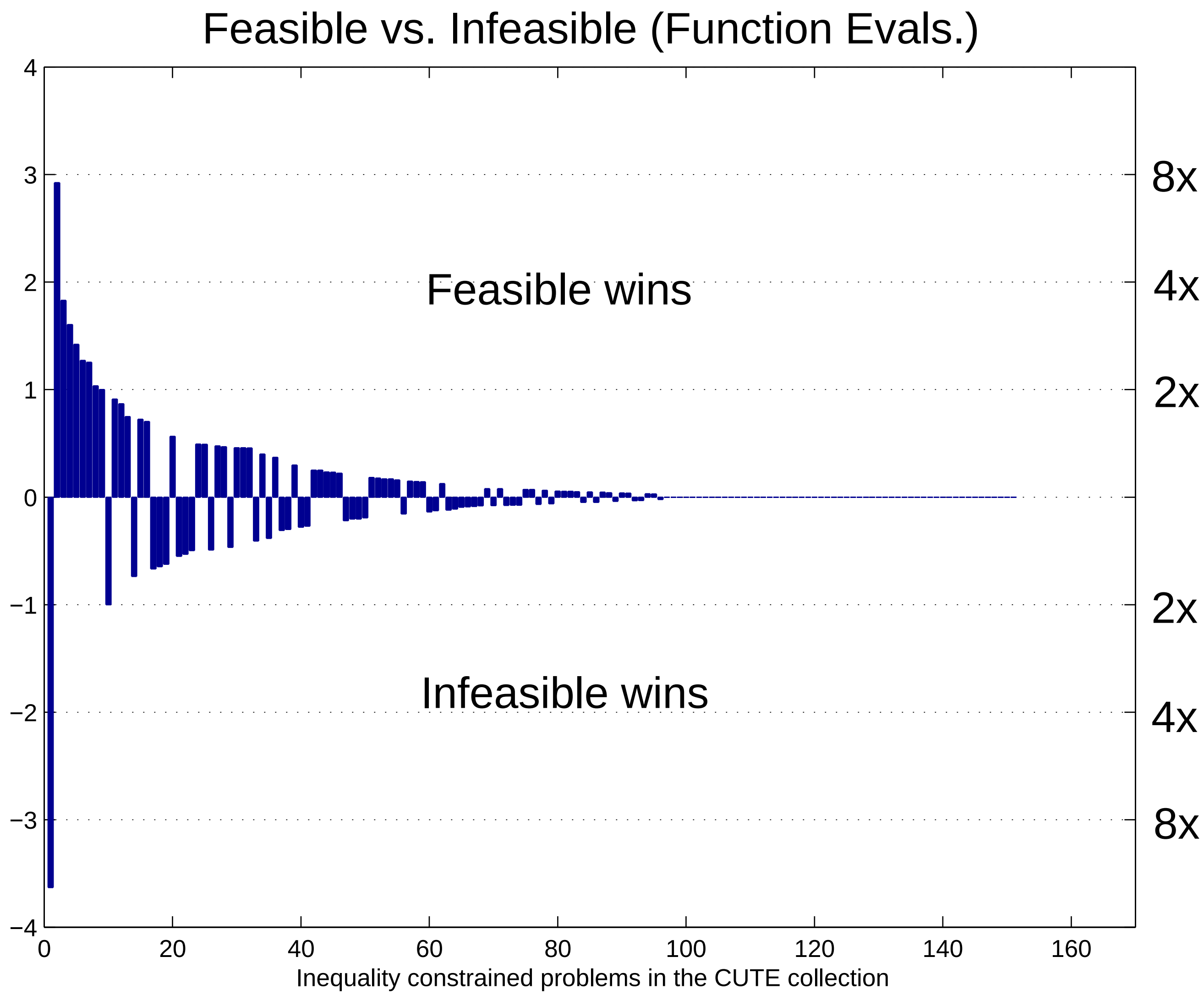
Newton on primal-dual equations of

$$\begin{aligned} \min_{x} \quad & \psi_{CB}(x;\mu) \equiv f(x) - \mu \sum_{i \in \mathcal{I}} \ln(g_i(x)) \\ \text{s.t.} \quad & h(x) = 0 \\ & g(x) > 0 \end{aligned}$$

equivalent to Newton on primal-dual equations of

$$\begin{aligned} \min_{x,s} \quad & \psi_{SB}(x,s;\mu) \equiv f(x) - \mu \sum_{i \in \mathcal{I}} \ln(s_i) \\ \text{s.t.} \quad & h(x) = 0 \\ & g(x) - s = 0 \\ & s > 0 \end{aligned}$$

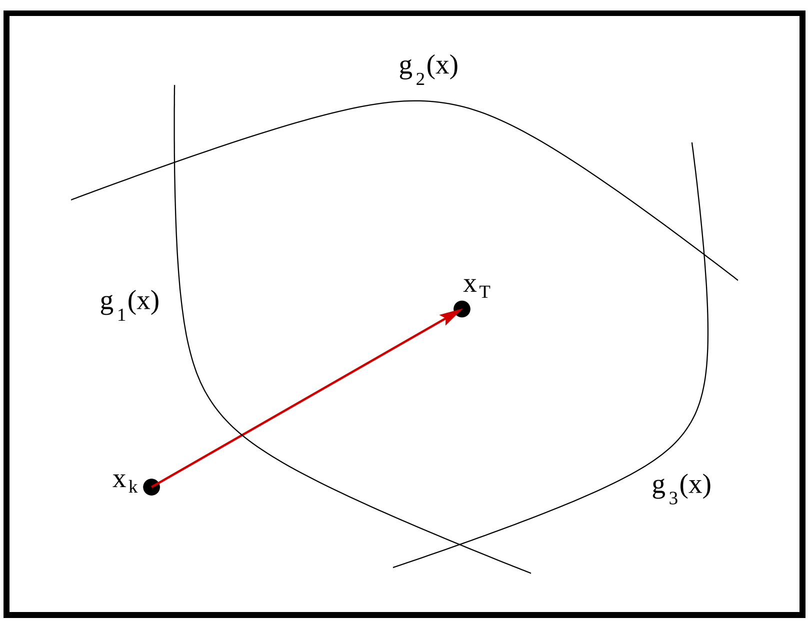
Numerical Results



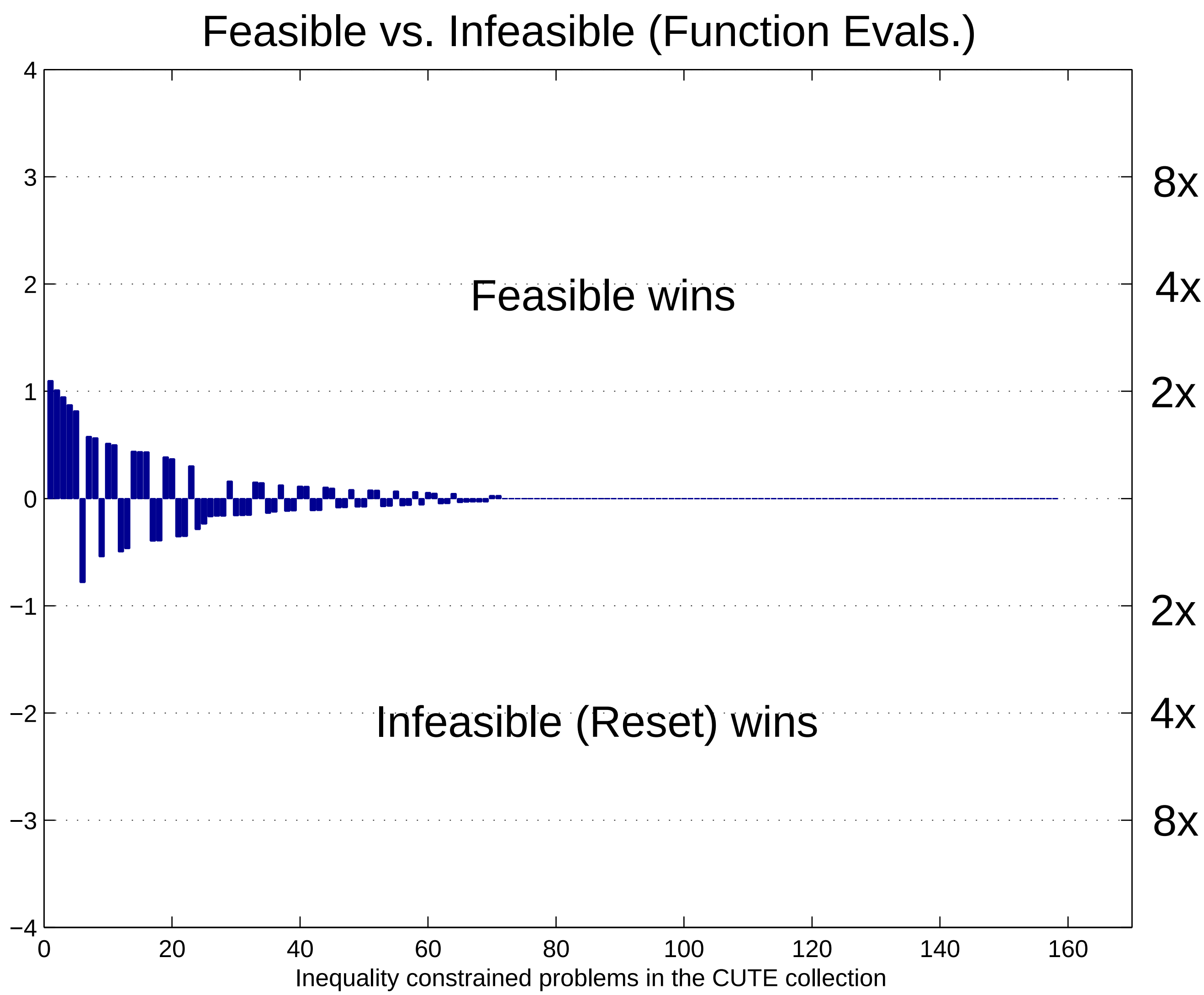
Infeasible Slack Reset Scheme

$$s_\top \leftarrow \max(g(x_\top), s + d_s)$$

- if $g(x_\top) > s + d_s$: feasible reset
  - go ahead and satisfy $g(x_\top) - s_\top = 0$
- if $g(x_\top) \leq s + d_s$: no reset
  (may take infeasible steps)
- only reset slacks if it will further decrease the merit function (can do no harm)



Numerical Results



Numerical Results

Robustness Comparison

Method	# of Problems Solved
Infeasible (NoReset)	170
Infeasible (Reset)	172
Feasible	174

- Test Set: 216 CUTE problems
- Not much difference

Summary/Conclusions

1. Slack-based feasible methods are derivable from infeasible methods.

   - LS: trivial modification

   - TR: requires *modified Cauchy step*

2. Slack-based feasible alg $\equiv$ classical method

3. You can improve the efficiency of an infeasible method by manipulating the slack variables.

4. Feasible vs. Infeasible?
   - Not a big difference overall
   - Feasible better on group of problems