Asymptotic Capacity of Beamforming with Limited Feedback[1]

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Abstract — **We study the channel capacity of a point-to-point communication system with multiple antennas and limited feedback. The receiver with perfect channel knowledge can relay B bits, which specify a beamforming vector, to the transmitter. We show that a Random Vector Quantization scheme is asymptotically optimal and give a simple expression for the associated capacity.**

I. Summary

We consider a flat Rayleigh fading channel with M transmit and N receive antennas. The received vector is given by $\boldsymbol{y} = \boldsymbol{H}\boldsymbol{v}x + \boldsymbol{w}$ where $\boldsymbol{H}$ is an $N \times M$ channel matrix whose element is a complex Gaussian random variable with zero mean and unit variance, $\boldsymbol{v}$ is an $M \times 1$ beamforming vector, x is a transmitted symbol with zero mean and unit variance. $\boldsymbol{y}$ is an $N \times 1$ received symbol vector, and $\boldsymbol{w}$ is an AWGN vector with covariance matrix $\frac{1}{\rho}\boldsymbol{I}$ where $\boldsymbol{I}$ is an identity matrix.

The mutual information $I(x, \boldsymbol{y})$ can be maximized over the beamforming vector $\boldsymbol{v}$ subject to a power constraint $\|\boldsymbol{v}\| \leq 1$. With B feedback bits, we can construct a vector quantizer with a codebook $\mathcal{V} = \{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_{2^B}\}$. Assuming that $\boldsymbol{H}$ is known at the receiver, it chooses $\boldsymbol{v}_j$ from $\mathcal{V}$ that maximizes $I(x, \boldsymbol{y})$, and relays the corresponding index back to the transmitter.

Optimizing the codebook $\mathcal{V}$ for finite M and N is quite difficult; however, as $(M, N) \to \infty$, the eigenvectors of $\boldsymbol{H}^\dagger \boldsymbol{H}$ are isotropically distributed. This suggests that the codebook entries should also be isotropically distributed. Hence in what follows, we assume that the vectors $\{\boldsymbol{v}_j\},\ j = 1, \cdots, 2^B$, are independent and randomly distributed with unit norm, and refer to this scheme as *Random Vector Quantization (RVQ)*. An analogous RVQ scheme has been previously proposed for CDMA signature optimization with limited feedback in [1] and has been shown to maximize the SINR in the large system limit. Our model is similar to that in [2], but our asymptotic approach based on RVQ differs substantially from prior work.

The receiver selects the quantized precoding vector $\hat{\boldsymbol{v}} = \arg\max_{1 \leq j \leq 2^B} \left\{ I_j = \log[1 + \rho\|\boldsymbol{H}\boldsymbol{v}_j\|^2] \right\}$ and the corresponding performance is given by

$$I_{\mathrm{rvq}}^M = E_{\boldsymbol{H},\mathcal{V}} \left[\max_{1 \leq j \leq 2^B} I_j \right] \qquad (1)$$

For single receive antenna ($N = 1$), $\boldsymbol{H}$ reduces to a channel vector. Evaluation of (1) for finite M and B is relatively difficult, so that we instead resort to evaluating a large system limit with fixed feedback bits per a transmit antenna.

Theorem 1 *As $(M, B) \to \infty$ with fixed $\bar{B} = B/M$,*

$$\mathcal{I}_{\mathrm{rvq}} = \lim_{(M,B)\to\infty} \left[I_{\mathrm{rvq}}^M - \log(\rho M) \right] = \log\left(1 - 2^{-\bar{B}}\right)$$

The proof relies on the asymptotic theory of extreme order statistics [3]. As $\bar{B} \to \infty$ (unlimited feedback), the capacity grows as $\log(\rho M)$. Hence this result says that for $\bar{B} > 0$, the asymptotic capacities with limited and unlimited feedback differ by a constant.

RVQ is asymptotically optimal in the following sense. Suppose that for each M, the elements of the vectors in the codebook $\mathcal{V}_M$ are selected from some arbitrary joint distribution $\mathcal{F}_{\mathcal{V}_M}$. Let $I_{\mathcal{F}_\mathcal{V}}^M$ denote the corresponding mutual information.

Theorem 2 *As $(M, B) \to \infty$ with fixed $\bar{B} = B/M$,*

$$\lim_{(M,B)\to\infty} \left[I_{\mathcal{F}_\mathcal{V}}^M - \log(\rho M) \right] \leq \mathcal{I}_{\mathrm{rvq}}$$

for any sequence of joint distributions $\{F_{\mathcal{V}_M}\}$.

For a system with *multiple* receive antennas ($N > 1$), we again consider an RVQ codebook. However, it is difficult to evaluate even an asymptotic performance in this case, so a lower bound is obtained instead. Similarly, RVQ is also an optimal quantization scheme here.

Theorem 3 *As $(M, N, B) \to \infty$ with fixed $\bar{N} = N/M$ and $\bar{B} = B/M$,*

$$\mathcal{I}_{\mathrm{rvq}} \geq \log\left(\left(1 + \sqrt{\bar{N}}\right)^2 \left(1 - 2^{-\bar{B}}\right) \right)$$

Furthermore, $\lim_{(M,N,B)\to\infty} \left[I_{\mathcal{F}_\mathcal{V}}^M - \log(\rho M) \right] \leq \mathcal{I}_{\mathrm{rvq}}$ *for any sequence of joint distributions $\{\mathcal{F}_{\mathcal{V}_M}\}$.*

The bound is tight when $\bar{B} = 0$ and $\bar{B} = \infty$ and gives a good estimate to a performance generated by simulation.

The optimal beamforming vector in the RVQ codebook must be found by exhaustive search. As a less complex, but suboptimal alternative, we consider a scalar quantizer with a *reduced-rank* (RR) beamformer. This scheme is motivated by an analogous reduced-rank signature optimization scheme [1]. That is, the beamforming vector $\boldsymbol{v}$ is constrained to lie in an R-dimensional subspace ($R \leq M$) and hence, fewer number of coefficients to be quantized. Each coefficient of the vector is quantized with the same quantizer and fed back to the transmitter. Numerical examples will be shown at the conference and illustrates that a quantized RR beamformer with optimized R performs close to the RVQ bound.

References

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