ISSSTA2004, Sydney, Australia, 30 Aug. - 2 Sep. 2004

Achievable Rates for MIMO Fading Channels With Limited Feedback and Linear Receivers

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***Abstract*— Channel information at the transmitter can simplify the coding scheme and increase the achievable data rate over a multiple-input multiple-output (MIMO) fading channel. Feedback from the receiver can be used to specify a precoding matrix, which selectively activates the strongest channel modes. We evaluate the sum data rate per receive antenna when the precoding matrix is quantized with a random vector quantization (RVQ) scheme, assuming a matched filter, or linear Minimum Mean Squared Error (MMSE) receiver. Our results are asymptotic as the number of transmit and receive antennas increases with fixed ratio, for a fixed number of feedback bits per dimension. Numerical results show that given a target spectral efficiency, the amount of feedback required by the linear MMSE receiver is only slightly more than that required by the optimal receiver, whereas the matched filter can require significantly more feedback. We also compare these results with a simpler reduced-rank scheme for quantizing the precoding matrix.**

I. INTRODUCTION

The performance of a communications channel generally depends on what channel information is available at the receiver and transmitter. For a multi-antenna, or Multi-Input/Multi-Output (MIMO) wireless channel with slow fading, relaying channel measurements at the receiver back to the transmitter can increase the capacity substantially when the Signal-to-Noise Ratio (SNR) is small and the ratio of transmitter to receiver antennas is large [1]. Perhaps more importantly, for a target information rate channel information at the transmitter can simplify the coding scheme.

Perfect channel knowledge at the transmitter can be used to select a precoding matrix, which activates a subset of eigenvectors of the channel covariance matrix. The eigenvectors represent a set of parallel Gaussian channels, so that standard coding and decoding schemes can be applied. The optimal receiver then consists of a bank of matched filters, and the filter outputs are the inputs to the decoder. Of course, in practice the channel information at the transmitter is unlikely to be perfect, and may be limited by the amount of feedback available. This leads to a suboptimal precoding matrix with associated interference among the matched filter outputs.

In this paper we evaluate the performance of limited feedback schemes for MIMO fading channels. Specifically, we assume that the precoding matrix is quantized with a finite number of bits, B. As originally proposed in [2], the receiver selects the transmit precoding matrix from a quantization set, or codebook, and relays the codebook index to the transmitter via a low rate channel. Here we evaluate the performance of a *random vector quantization (RVQ)* scheme in which the codebook for the precoding matrix consists of independent partial unitary matrices. RVQ has been previously analyzed with an optimal receiver in [1], and is asymptotically optimal (maximizes capacity) with a Multi-Input/Single-Output (MISO) channel [3]. (In that case RVQ is used to quantize a transmit beamforming vector.) Here we consider RVQ with simple linear receivers, i.e., matched filter and Minimum Mean Square Error (MMSE) receivers.

With sufficient feedback the linear receivers are optimal, since the precoding and receiver matrices diagonalize the channel. We are therefore interested in determining how much more feedback is needed by these receivers to achieve the same performance as the optimal receiver. Our results are asymptotic as the number of transmit and receive antennas tend to infinity, given a fixed number of feedback bits per dimension (number of transmit times receive antennas). A numerical example shows that given a target spectral efficiency, the linear MMSE receiver with RVQ requires only slightly more feedback than the optimal receiver. The matched filter requires significantly more feedback (in the range of 0.4-0.8 bit/dimension) for the example shown. The numerical results also show that the large system analysis provides a good estimate for the performance of finite size systems of interest.

Related work on limited feedback schemes for MIMO and MISO channels is presented in [3]–[7]. Vector quantization schemes based on Grassmannian, or line packing are considered in [4]–[6]. The performance of the optimal vector quantization codebook generated by the Lloyd-Max algorithm is approximated in [7]. Our focus here is on the performance of RVQ, which can be evaluated in the large system limit.

Although generating an RVQ codebook is simple, selecting the codebook entry that optimizes performance generally requires an exhaustive search over all codebook entries. The complexity therefore grows exponentially with the number of feedback bits. We therefore consider a simpler *reduced-rank* quantization scheme in which the receiver quantizes the precoding matrix column-by-column. Each column is con-

This work was supported by the U.S. Army Research Office under grant DAAD190310119 and the National Science Foundation under grant CCR-0310809.

strained to lie in a lower dimensional subspace. The receiver optimizes each column within the subspace, and relays the scalar-quantized coefficients back to the transmitter. Numerical results show that this scheme requires much less feedback than scalar quantization of all precoding matrix coefficients, but still requires substantially more feedback than RVQ.

II. System Model

We consider a single-user, flat Rayleigh fading channel with M transmit antennas and N receive antennas. The channel coefficient $h_{n,m}$ between the mth transmit antenna and the nth receive antenna is a complex Gaussian random variable with zero mean and unit variance ($E[|h_{n,m}|^2] = 1$), and the set of channel coefficients are independent identically distributed (i.i.d.). The received N-vector is given by

$$\boldsymbol{y} = \frac{1}{\sqrt{K}} \boldsymbol{H}\boldsymbol{V}\boldsymbol{x} + \boldsymbol{w} \tag{1}$$

where $\boldsymbol{H} = [h_{n,m}]$ is an $N \times M$ channel matrix, $\boldsymbol{V} = [\boldsymbol{v}_1 \; \boldsymbol{v}_2 \; \ldots \; \boldsymbol{v}_K]$ is an $M \times K$ precoding matrix, $\boldsymbol{x} = [x_k]$ is a K-vector of transmitted symbols, $\boldsymbol{w}$ is a complex Gaussian noise N-vector with covariance matrix $\sigma_n^2 \boldsymbol{I}$, where $\boldsymbol{I}$ is an identity matrix, and K is the number of data streams.

The symbols corresponding to data stream k are recovered by passing the received signal $\boldsymbol{y}$ through the filter (N-vector) $\boldsymbol{c}_k$. The matched filter is given by

$$\boldsymbol{c}_k = \frac{1}{\sqrt{K}} \boldsymbol{H}\boldsymbol{v}_k \tag{2}$$

and the linear MMSE filter is given by

$$\boldsymbol{c}_k = \frac{1}{\sqrt{K}} \left(\frac{1}{K} \boldsymbol{H}\boldsymbol{V}\boldsymbol{V}^\dagger \boldsymbol{H}^\dagger + \sigma_n^2 \boldsymbol{I} \right)^{-1} \boldsymbol{H}\boldsymbol{v}_k \tag{3}$$

The signal-to-interference plus noise ratio (SINR) at the output of the linear filter $\boldsymbol{c}_k$ is

$$\mathrm{SINR}_k = \frac{\left(\boldsymbol{c}_k^\dagger \boldsymbol{H}\boldsymbol{v}_k \right)^2}{\boldsymbol{c}_k^\dagger \left(\sum_{i \neq k} \boldsymbol{H}\boldsymbol{v}_i \boldsymbol{v}_i^\dagger \boldsymbol{H}^\dagger + K\sigma_n^2 \boldsymbol{I} \right) \boldsymbol{c}_k} \tag{4}$$

The filter outputs are then passed to the decoder. In what follows, we assume separate coders and decoders for each data stream with independent interleavers and de-interleavers, which reduces the correlation among interference terms at the outputs of the receiver filters.

The performance measure is mutual information between the transmitted symbol x_k and the output of the filter $\boldsymbol{c}_k$, denoted as $\hat{x}_k$. Assuming that the interference plus noise at the output of the linear filter has a Gaussian distribution, which is true in the large system limit to be considered, the sum mutual information of all data streams per receive antenna is given by

$$R = \frac{1}{N} \sum_{k=1}^{K} I(x_k, \hat{x}_k) \tag{5}$$

$$= \frac{1}{N} \sum_{k=1}^{K} \log(1 + \mathrm{SINR}_k) \tag{6}$$

Given a channel matrix $\boldsymbol{H}$, the instantaneous sum rate R depends on the precoding matrix $\boldsymbol{V}$. We are interested in maximizing the sum mutual information subject to the power constraint $\|\boldsymbol{v}_k\| \leq 1, \; \forall k$. We assume that the power is allocated equally across streams. The optimal water-filling power allocation requires additional feedback. Our numerical results, to be presented, show that this additional optimization gives only a marginal increase in mutual information.

We also assume that the feedback channel is error free and incurs no delay. The system performance therefore depends only on the feedback rate and the codebook.

III. Random Vector Quantization

Given B feedback bits, we can construct a codebook with 2^B entries for the precoding matrix, which is denoted as

$$\mathcal{V} = \{\boldsymbol{V}_j, \; 1 \leq j \leq 2^B\} \tag{7}$$

This codebook is known *a priori* at both the transmitter and receiver. Given the channel matrix $\boldsymbol{H}$, the receiver selects

$$\hat{\boldsymbol{V}} = \arg \max_{1 \leq j \leq 2^B} R(\boldsymbol{V}_j) \tag{8}$$

and relays the corresponding index to the transmitter.

The design of an optimal (rate-maximizing) codebook depends on the distribution of the channel matrix, and is a difficult open problem for a general set of parameters (N, M, B, K). References [4], [6] characterize the optimal codebook for a rank-one precoding matrix (beamformer). Also, the optimal performance is known for the extreme cases $B = 0$ (no feedback), and $B \to \infty$ (unlimited feedback). When $B = 0$, a random $M \times K$ unitary precoding matrix $\boldsymbol{V}$ maximizes the ergodic capacity in the large system limit (to be defined). With unlimited feedback the columns of the optimal precoding matrix are eigenvectors of the channel covariance matrix $\boldsymbol{H}^\dagger \boldsymbol{H}$.

As $(N, M) \to \infty$, the eigenvectors of $\boldsymbol{H}^\dagger \boldsymbol{H}$ are random and isotropically distributed. This suggests that the codebook entries should also be isotropically distributed. Hence in what follows, we take the entries of the codebook $\mathcal{V}$ to be a set of independent random partial unitary matrices, and refer to this scheme as *Random Vector Quantization (RVQ)*. RVQ is asymptotically optimal (maximizes mutual information) when used to quantize a transmit beamforming vector for a MISO channel [3]. Whether or not it is asymptotically optimal for the MIMO channel considered is an open question.

The sum mutual information per receive antenna with RVQ can be computed as follows

$$R_{\mathrm{rvq}} = E_{\boldsymbol{H},\mathcal{V}} \left[\max_{1 \leq j \leq 2^B} R(\boldsymbol{V}_j) \right] \tag{9}$$

where the expectation is over the channel realization and codebook. Since the codebook entries are i.i.d., the corresponding set of rates $\{R(\boldsymbol{V}_j)\}$ are also i.i.d. with associated probability density function (pdf) f_R and cumulative distribution function (cdf) F_R. We can therefore rewrite (9) as

$$R_{\mathrm{rvq}} = 2^B \int_0^\infty x[F_R(x)]^{2^B - 1} f_R(x) \, \mathrm{d}x \tag{10}$$

This is difficult to evaluate with a finite number of feedback bits. However, as $(B, N, M, K) \to \infty$ with fixed $\bar{N} = N/M$, $\bar{K} = K/N$, and $\bar{B} = B/N^2$, the distribution of $\max_j R(\boldsymbol{V}_j)$ with proper shifting and scaling converges to a Gumbel distribution [8, Theorem 2.1.3]. The asymptotic average sum mutual information is given by

$$\mathcal{R}_{\rm rvq} = \lim_{(B,N,M,K)\to\infty} R_{\rm rvq} \tag{11}$$

$$= \lim_{(B,N,M,K)\to\infty} F_R^{-1}\left(1 - \frac{1}{2^B}\right) \tag{12}$$

This expression is the same as that for the optimal receiver considered in [1], and is derived using the theory of extreme order statistics [8]. Of course, the cdf F_R depends on the particular receiver.

A. Matched filter

Substituting (2) into (4), the SINR at the output of matched filter is given by

$$\gamma_k = \left(\frac{K\sigma_n^2}{\boldsymbol{v}_k^\dagger \boldsymbol{H}^\dagger \boldsymbol{H} \boldsymbol{v}_k} + \frac{\sum_{i=1, i\neq k}^{K} \left(\boldsymbol{v}_k^\dagger \boldsymbol{H}^\dagger \boldsymbol{H} \boldsymbol{v}_i\right)^2}{\left(\boldsymbol{v}_k^\dagger \boldsymbol{H}^\dagger \boldsymbol{H} \boldsymbol{v}_k\right)^2} \right)^{-1} \tag{13}$$

and the sum rate per receive antenna is

$$R_{\rm mf} = \frac{1}{N} \sum_{k=1}^{K} \log(1+\gamma_k). \tag{14}$$

For a finite-size system it is difficult to obtain a closed-form expression for the pdf of the SINR, which can be used to obtain the pdf of $R_{\rm mf}$. As the system size increases, so does the number of terms in the sum mutual information, so that the central limit theorem implies

$$N\left(R_{\rm mf} - \mu_{\rm mf}\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \sigma_{\rm mf}^2\right) \tag{15}$$

as $(N, M, K) \to \infty$. We therefore approximate the cdf of $R_{\rm mf}$ for finite (N, M, K) with a Gaussian cdf with mean $\mu_{\rm mf}$ and variance $\sigma_{\rm mf}^2/N^2$. The limit in (12) then evaluates to [1]

$$\mathcal{R}_{\rm mf} \approx \tilde{\mathcal{R}}_{\rm mf} = \mu_{\rm mf} + \sigma_{\rm mf}\sqrt{2\bar{B}\log 2} \tag{16}$$

where

$$\mu_{\rm mf} = \bar{K} \log\left(1 + \frac{1}{\bar{K}(1+\sigma_n^2)}\right), \tag{17}$$

and follows from the fact that γ_k converges almost surely to $[\bar{K}(1+\sigma_n^2)]^{-1}$ as $(N, M, K) \to \infty$.

To evaluate $\sigma_{\rm mf}^2$, we first compute the variance of $\log(1+\gamma_k)$. Letting $Z = 1/\gamma_k$, as $(N, M, K) \to \infty$ with fixed $\bar{N}$ and $\bar{K}$, we can compute

$$N{\rm var}[Z] \longrightarrow \bar{K} + \bar{K}^2\left(1 + 6\sigma_n^2 + 3\sigma_n^4\right) \tag{18}$$

We note that this follows from the limits

$$\frac{1}{N}{\rm var}\left[\boldsymbol{v}_k^\dagger \boldsymbol{H}^\dagger \boldsymbol{H} \boldsymbol{v}_k\right] \longrightarrow 1 \tag{19}$$

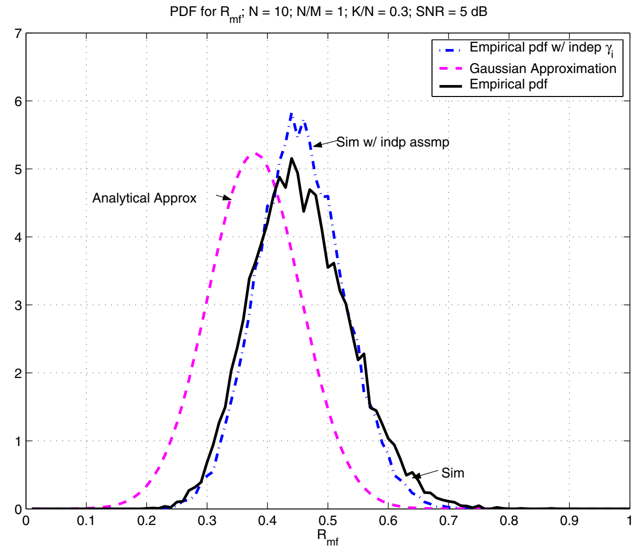

Fig. 1. Comparison of the empirical pdf for $R_{\rm mf}$ with the asymptotic Gaussian approximation.

and

$$\frac{1}{N}{\rm var}\left[\sum_{i=1, i\neq k}^{K} \left(\boldsymbol{v}_k^\dagger \boldsymbol{H}^\dagger \boldsymbol{H} \boldsymbol{v}_i\right)^2\right] \longrightarrow \bar{K} \tag{20}$$

From (18), after some algebraic manipulations we obtain

$$N{\rm var}\left[\log(1+\gamma_k)\right] \to \frac{1 + \bar{K}\left(1 + 6\sigma_n^2 + 3\sigma_n^4\right)}{\bar{K}^3(1+\sigma_n^2)^2(1+\sigma_n^2+1/\bar{K})^2} \tag{21}$$

Turning to the variance of $R_{\rm mf}$, we must evaluate the cross-correlations between terms in the sum in (14). For small $\bar{K}$, the sum of all cross-terms in the variance calculations is small, and the variance of $R_{\rm mf}$ can be accurately approximated as $\bar{K}{\rm var}\left[\log(1+\gamma_k)\right]/N$, i.e.,

$$\sigma_{\rm mf}^2 \approx \frac{1 + \bar{K}\left(1 + 6\sigma_n^2 + 3\sigma_n^4\right)}{\bar{K}^2(1+\sigma_n^2)^2(1+\sigma_n^2+1/\bar{K})^2} \tag{22}$$

Fig. 1 compares an empirical pdf for $R_{\rm mf}$ with the Gaussian approximation for $N = 10$, $\bar{N} = 1$, $\bar{K} = 0.3$ and SNR = 5 dB. We also plot the empirical pdf for $R_{\rm mf}$ when the SINR's γ_k, $k = 1, \cdots, K$, are generated from independent channels, so that the terms in (14) are independent. The corresponding pdf is quite close to the empirical pdf of $R_{\rm mf}$, which justifies the independence assumption. Also, the estimated variance is quite close to the empirical value. The difference between the empirical and asymptotic means vanishes as $(M, N, K) \to \infty$. We remark, however, that the independence approximation becomes less accurate as $\bar{K}$ increases.

The asymptotic approximation for RVQ performance in (16) depends on only $\bar{K}$, $\bar{B}$, and σ_n^2. This approximation is tight as $\bar{B} \to 0$; however, $\tilde{\mathcal{R}}_{\rm mf} \to \infty$ as $\bar{B} \to \infty$, whereas the actual rate with unlimited feedback is finite. This is because the Gaussian distribution has infinite support, whereas F_R must have finite support. With unlimited feedback the columns of the optimal precoding matrix are the

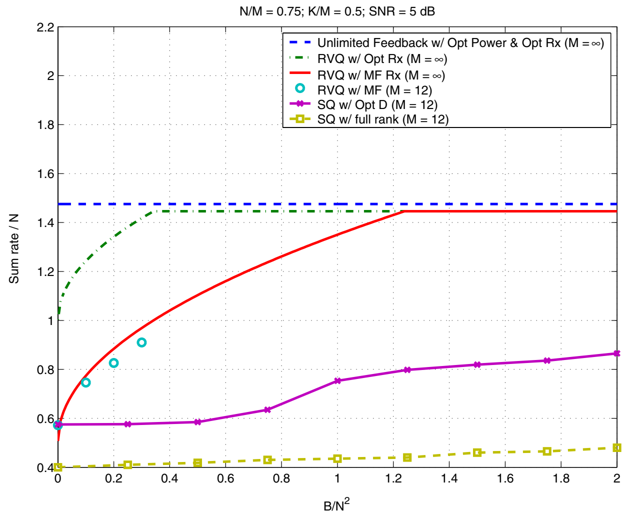

Fig. 2. Sum rate per receive antenna versus normalized feedback bits with the matched filter receiver.

eigenvectors corresponding to the K largest eigenvalues of the channel covariance matrix $\boldsymbol{H}^\dagger\boldsymbol{H}/M$. Since the eigenvalue distribution for $\boldsymbol{H}^\dagger\boldsymbol{H}/M$ converges to a deterministic function as $(N, M) \to \infty$, we have

$$\mathcal{R}_{\mathrm{mf}}(\bar{B}=\infty) = \frac{1}{\bar{N}}\int_c^b \log\left(1+\frac{1}{\bar{K}\sigma_n^2}\lambda\right) f(\lambda)\,\mathrm{d}\lambda \quad (23)$$

where the asymptotic eigenvalue density is

$$f(\lambda) = (1-\bar{N})^+\delta(\lambda) + \frac{\bar{N}}{2\pi\lambda}\sqrt{(\lambda-a)(b-\lambda)} \quad (24)$$

where

$$a = \left(1-\frac{1}{\sqrt{\bar{N}}}\right)^2 \quad \text{and} \quad b = \left(1+\frac{1}{\sqrt{\bar{N}}}\right)^2 \quad (25)$$

and c satisfies

$$1-\bar{K}\bar{N} = \int_0^c f(\lambda)\,\mathrm{d}\lambda \quad (26)$$

Fig. 2 compares the approximation for asymptotic RVQ performance with a matched filter receiver from (16) with simulated results for $M=12$, $\bar{N}=0.75$, $\bar{K}=2/3$, and SNR $=$ 5 dB. For the case shown, the analytical approximation gives an accurate estimate for the performance of the finite size system with limited feedback. The asymptotic performance of RVQ with an optimal receiver [1] is also shown along with the capacity with the optimal water pouring power allocation ($\bar{B}=\infty$). The optimal receiver requires approximately $\bar{B}=$ 0.4 bit/dimension to achieve the capacity corresponding to unlimited feedback (23), whereas the matched filter requires 1.2 feedback bits per dimension to reach that capacity. The capacity with the water-filling power allocation is only slightly greater than that achieved with an equal power allocation.

B. MMSE receiver

To compute the asymptotic RVQ performance with the MMSE receiver, we must again compute the corresponding pdf F_R for the sum mutual information. Substituting (3) into (4) gives the output SINR for the kth symbol stream

$$\beta_k = \boldsymbol{v}_k^\dagger\boldsymbol{H}^\dagger\left(\sum_{i=1,i\neq k}^{K}\boldsymbol{H}\boldsymbol{v}_i\boldsymbol{v}_i^\dagger\boldsymbol{H}^\dagger + K\sigma_n^2\boldsymbol{I}\right)^{-1}\boldsymbol{H}\boldsymbol{v}_k \quad (27)$$

The distribution of β_k converges to a Gaussian distribution [9]

$$\sqrt{N}(\beta_k-\beta) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0,\sigma_\beta^2\right) \quad (28)$$

where

$$\beta = \frac{1-\bar{K}}{2\sigma_n^2} - \frac{1}{2} + \sqrt{\frac{(1-\bar{K})^2}{4\sigma_n^4} + \frac{1+\bar{K}}{2\sigma_n^2} + \frac{1}{4}} \quad (29)$$

and

$$\sigma_\beta^2 = \frac{2\beta(1+\beta)^2}{\sigma_n^2(1+\beta)^2+\bar{K}} - \beta^2 \quad (30)$$

Hence $N\mathrm{var}\left[\beta_k\right]$ converges to σ_β^2 almost surely as $(N, M, K) \to \infty$, and we can proceed to compute the limit of $N\mathrm{var}\left[\log(1+\beta_k)\right]$. We again assume that the asymptotic sum rate is Gaussian, i.e.,

$$N\left(R_{\mathrm{mmse}}-\mu_{\mathrm{mmse}}\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0,\sigma_{\mathrm{mmse}}^2\right) \quad (31)$$

where R_{mmse} is the sum rate per receive antenna with an MMSE receiver,

$$\mu_{\mathrm{mmse}} = \bar{K}\log(1+\beta) \quad (32)$$

and

$$\sigma_{\mathrm{mmse}}^2 \approx \frac{\bar{K}\sigma_\beta^2}{(1+\beta)^2}. \quad (33)$$

This expression again ignores the cross correlations between the SINR's, which are relatively small for small to moderate values of $\bar{K}$. In analogy with (16), the asymptotic rate for RVQ with the MMSE receiver is given by

$$\mathcal{R}_{\mathrm{mmse}} \approx \tilde{\mathcal{R}}_{\mathrm{mmse}} = \mu_{\mathrm{mmse}} + \sigma_{\mathrm{mmse}}\sqrt{2\bar{B}\log 2} \quad (34)$$

As for the matched filter receiver, when $\bar{B}$ is large, $\tilde{\mathcal{R}}_{\mathrm{mmse}}$ over-estimates $\mathcal{R}_{\mathrm{mmse}}$. For $\bar{B}=\infty$, $\mathcal{R}_{\mathrm{mmse}} = \mathcal{R}_{\mathrm{mf}}$, which can be computed from (23).

Fig. 3 compares the asymptotic approximation for the performance of RVQ with an MMSE receiver with simulation results. These results show that for the parameters selected, the MMSE receiver performs nearly as well as the optimal receiver, and requires substantially less feedback than the matched filter to achieve a target rate. We also observe that the asymptotic approximation gives a more accurate performance prediction for the matched filter than for the MMSE filter with a relatively small number of antennas.

To illustrate further the accuracy of the approximate asymptotic results, Fig. 4 shows simulated and asymptotic sum rate per receive antenna versus $\bar{K}$ with $\bar{B}=0.2$, $\bar{N}=0.5$, and SNR $=$ 5 dB. Results are again shown for the linear

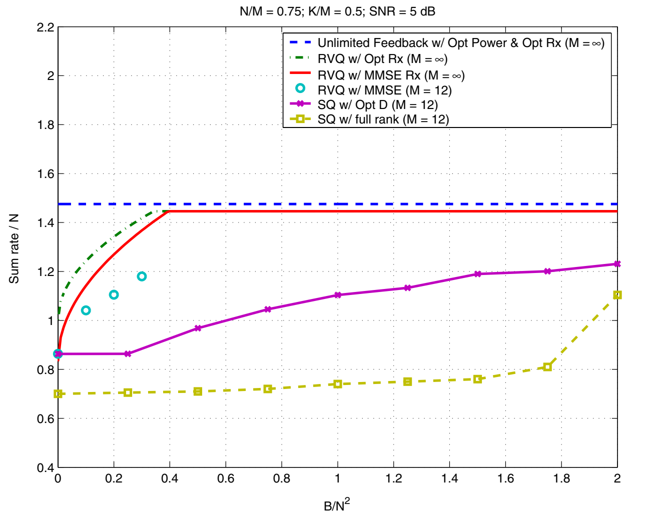

Fig. 3. Sum rate per receive antenna versus normalized feedback bits with a linear MMSE receiver.

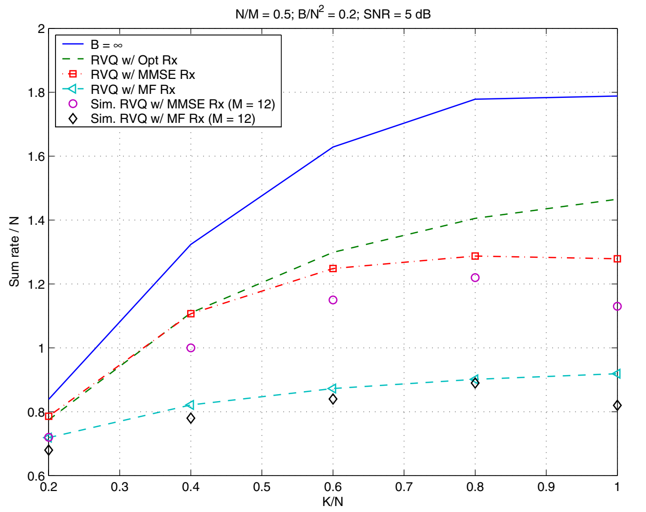

Fig. 4. Sum rate per receive antenna versus $\bar{K}$.

receivers and the optimal receiver, and for infinite feedback with water filling. The asymptotic results accurately predict the simulated results for $\bar{K} \leq 0.8$, and become less accurate as $\bar{K}$ approaches one. The MMSE receiver performs close to the optimal receiver for small $\bar{K}$ since there are fewer active channel modes, and therefore less interference.

IV. Reduced-Rank Scalar Quantization

RVQ requires that for each channel realization, the receiver select the precoding matrix that maximizes the sum mutual information. In general, this must be accomplished by exhaustive search, which means the complexity of this scheme increases exponentially with the number of feedback bits. We are therefore interested in simpler suboptimal schemes, which perform better than independent scalar quantization of each entry in the precoding matrix.

Here we apply the reduced-rank signature optimization scheme proposed in [10] for CDMA to selection of the precoding matrix. (This scheme was applied to quantization of a beamforming vector for a MISO system in [3].) Namely, we constrain each $M \times 1$ column of the precoding matrix, $\boldsymbol{v}_k$, $k = 1, \cdots, K$, to lie in a D-dimensional subspace ($D \leq M$). We therefore have

$$\boldsymbol{v}_k = \boldsymbol{F}_k \boldsymbol{\alpha}_k \tag{35}$$

where $\boldsymbol{F}_k$ is an $M \times D$ matrix of orthogonal basis vectors (known to the transmitter) and $\boldsymbol{\alpha}_k$ is a D-vector of combining coefficients. The DK coefficients in $\boldsymbol{\alpha}_1, \cdots, \boldsymbol{\alpha}_K$ are then *scalar*-quantized, and relayed to the transmitter. For the MISO channel with limited feedback, the performance of this scheme is comparable to RVQ [3].

For the MIMO channel the performance depends on the set $\mathcal{F} = \{\boldsymbol{F}_k, \ 1 \leq k \leq K\}$. We select each $\boldsymbol{F}_k$ to be a random partial unitary matrix ($\boldsymbol{F}_k^\dagger \boldsymbol{F}_k = \boldsymbol{I}$). If $KD \leq M$, we can design the set such that the subspaces associated with each column of the precoding matrix are orthogonal, i.e., $\boldsymbol{F}_k^\dagger \boldsymbol{F}_l = 0$ for all $k \neq l$. If $KD > M$, then we group all matrices in $\mathcal{F}$ into $\lceil KD/M \rceil$ subsets with at most $\lfloor M/D \rfloor$ matrices in each subset. Within each subset, the $\lfloor M/D \rfloor$ subspaces spanned by the columns of each of the matrix elements are orthogonal. This is accomplished by taking the columns of each $\boldsymbol{F}_k$ within a subset from a randomly generated unitary matrix.

To compute the set of $\boldsymbol{\alpha}_k$'s, which maximizes the sum instantaneous mutual information, we follow the iterative algorithm in [10]. Namely, for the MMSE receiver, we replace $\boldsymbol{\alpha}_k$ with the eigenvector of $\boldsymbol{F}_k^\dagger \boldsymbol{H}^\dagger \left(\sum_{i \neq k} \boldsymbol{H} \boldsymbol{v}_i \boldsymbol{v}_i^\dagger \boldsymbol{H}^\dagger + K\sigma_n^2 \boldsymbol{I} \right)^{-1} \boldsymbol{H} \boldsymbol{F}_k$ corresponding to the maximum eigenvalue. We do this for $k = 1$ to $k = K$, compute R_{mmse}, and iterate until R_{mmse} converges.

For the matched filter, computing the jointly optimal $\boldsymbol{\alpha}_k$'s is more involved [10]. Here we construct a near-optimal set, which after quantization, gives essentially the same performance. Namely, we approximate the optimization step in [10] by replacing each $\boldsymbol{\alpha}_k$, $k = 1, \cdots, K$, with the eigenvector of $\boldsymbol{F}_k^\dagger \boldsymbol{H}^\dagger \left(\sum_{i \neq k} \boldsymbol{H} \boldsymbol{v}_i \boldsymbol{v}_i^\dagger \boldsymbol{H}^\dagger \right) \boldsymbol{H} \boldsymbol{F}_k$ corresponding to the minimum eigenvalue. As with the MMSE receiver, we iterate until R_{mf} converges.

Each real and imaginary part of the elements of $\boldsymbol{\alpha}_k$ is quantized with $B/(2DK)$ feedback bits. The Lloyd-Max quantizer [11] is used, where we assume that the distribution for both the real and imaginary parts is Gaussian with zero mean and variance $1/(2D)$. Numerical comparisons with empirical pdf's have shown that this is a reasonable assumption.

The performance of limited feedback with a reduced-rank scalar quantizer is shown in Figs. 2 and 3, corresponding to matched filter and MMSE receivers, respectively. For a given feedback rate $\bar{B}$, the performance can be optimized over the rank D. The performance with a full-rank scalar quantizer ($D = M$) is also shown for comparison. With the MMSE receiver the achievable rate with the reduced-rank quantizer is about midway between the achievable rate for RVQ and the full-rank scalar quantizer. The reduced-rank scalar quantizer offers less of a performance gain (relative to the full-rank scalar quantizer) with the matched filter. The performance gains shown here are significantly less than those observed for the MISO channel [3]. This is because the subspaces spanned by the different $\boldsymbol{F}_k$'s can overlap substantially, which highly correlate the columns of the quantized precoding matrix.

V. Conclusions

We have studied the performance (i.e., achievable rate) of a MIMO channel with limited feedback and linear receivers. The RVQ scheme presented is motivated by its asymptotic optimality for the MISO channel [3]. With sufficient feedback the matched filter and linear MMSE receivers are optimal, hence our asymptotic results show how much additional feedback is needed by these simpler receivers to achieve a target rate, relative to the optimal receiver. For the numerical example shown here the MMSE receiver requires very little additional feedback, whereas the matched filter requires significantly more feedback. Specifically, to achieve the rate with perfect channel knowledge, both the MMSE and optimal receivers require about 0.4 bit/dimension, whereas the matched filter requires about 1.2 bits/dimension. The analytical approximations accurately predict the simulation results when the ratio of number of data streams to receive antennas is significantly less than one. The reduced-rank scalar quantizer presented can provide a substantial increase in rate relative to full-rank scalar quantization; however, there is still a substantial gap between reduced-rank and RVQ performance, unlike the MISO case [3].

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